Why the IRS cares about the Riemann Zeta Function and Number Theory (and why you should too!)

Steven J. Miller
sjm1@williams.edu,
Steven.Miller.MC.96@aya.yale.edu

http://web.williams.edu/Mathematics/
sjmiller/public_html/

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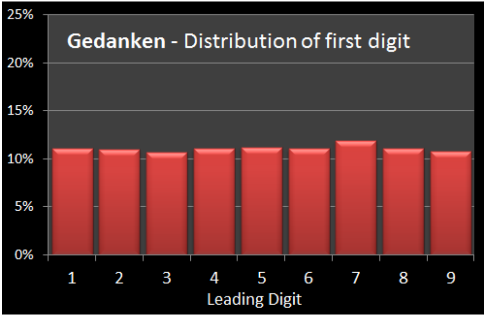
Introduction

- A. Berger and T. P. Hill, *An Introduction to Benford's Law*, Princeton University Press, Princeton, 2015. See also http://www.benfordonline.net/.
- A. E. Kossovsky, *Benford's Law: Theory, the General Law of Relative Quantities, and Forensic Fraud Detection Applications*, WSPC, 2014.
- S. J. Miller (editor), *Theory and Applications of Benford's Law*, Princeton University Press, 2015.
- M. Nigrini, *Benford's Law: Applications for Forensic Accounting, Auditing, and Fraud Detection*, 1st Edition, Wiley, 2014.

Interesting Question

Motivating Question: For a nice data set, such as the Fibonacci numbers, stock prices, street addresses of college employees and students, ..., what percent of the leading digits are 1?



Natural guess: 10% (but immediately correct to 11%!).

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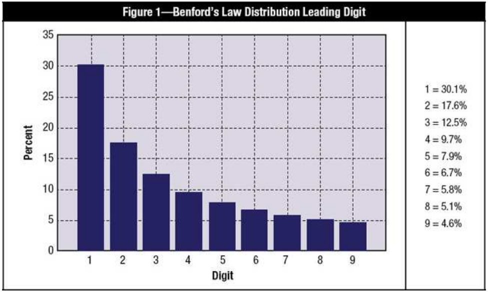
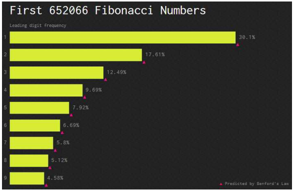


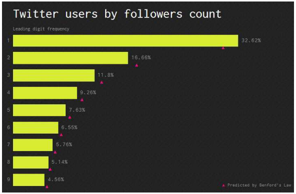
Answer: Benford's law!

Examples with First Digit Bias

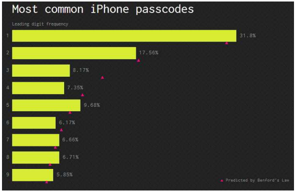
Fibonacci numbers



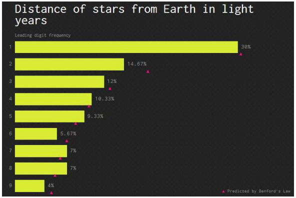
Twitter users by # followers



Most common iPhone passcodes



Distance of stars from Earth



Summary

- Explain Benford's Law.
- Discuss examples and applications.
- Sketch proofs.
- Describe open problems.

Caveats!

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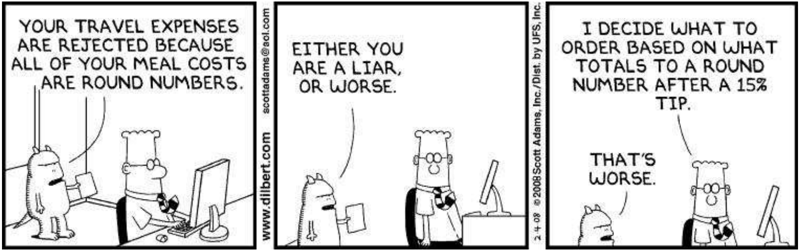


Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- L-functions, characteristic polynomials
- iterates of the $3x + 1$ map
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models

Applications

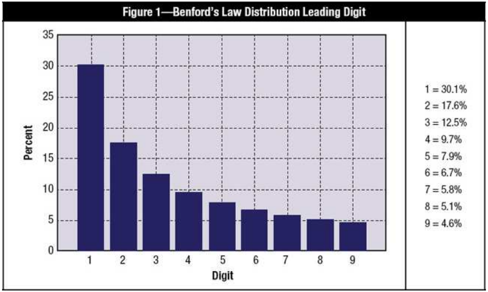
- Analyzing round-off errors.
- Determining the optimal way to store numbers.
- Detecting tax and image fraud, and data integrity.

General Theory

Benford's Law: Newcomb (1881), Benford (1938)

Statement

For many data sets, probability of observing a first digit of d base B is $\log_B\left(\frac{d+1}{d}\right)$; base 10 about 30% are 1s.



Benford's Law (probabilities)

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- Key observation: $\log_{10}(x) = \log_{10}(\widetilde{x}) \mod 1$ if and only if x and $\widetilde{x}$ have the same leading digits.

Thus often study $y = \log_{10} x \mod 1$.
Advanced: $e^{2\pi i u} = e^{2\pi i (u \mod 1)}$.

Equidistribution and Benford's Law

Equidistribution

$\{y_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if probability $y_n \bmod 1 \in [a, b]$ tends to $b - a$:

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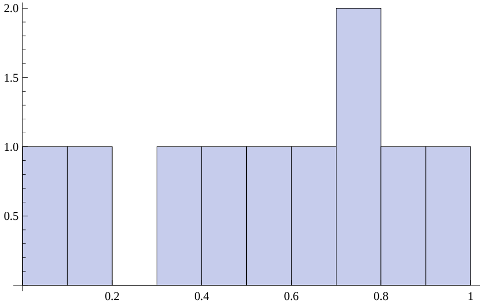
  Proof: if rational: $2 = 10^{p/q}$.
  Thus $2^q = 10^p$ or $2^{q-p} = 5^p$, impossible.

Example of Equidistribution: $n\sqrt{\pi} \bmod 1$



$n\sqrt{\pi} \bmod 1$ for $n \leq 10$

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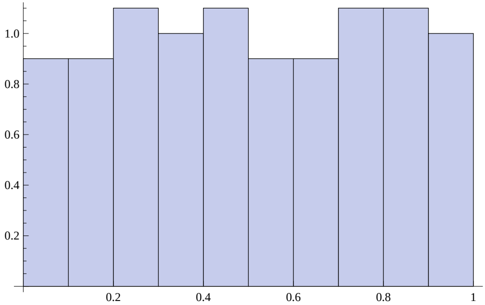

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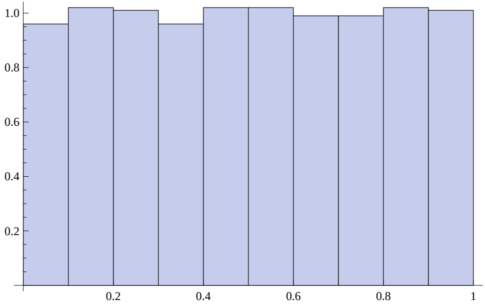
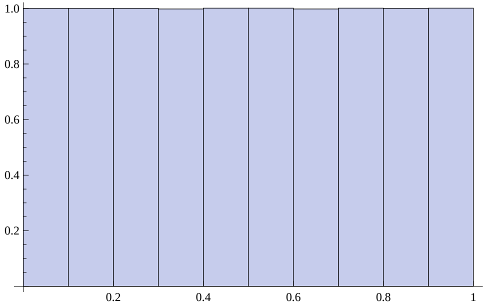


$n\sqrt{\pi} \bmod 1$ for $n \leq 1000$

Example of Equidistribution: $n\sqrt{\pi}$ mod 1



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Logarithms and Benford's Law

Fundamental Equivalence

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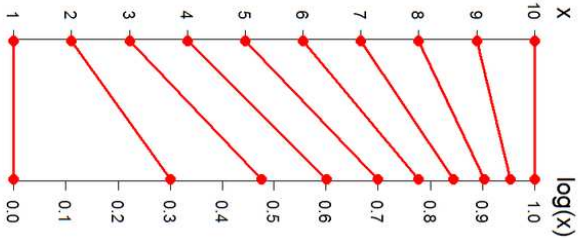
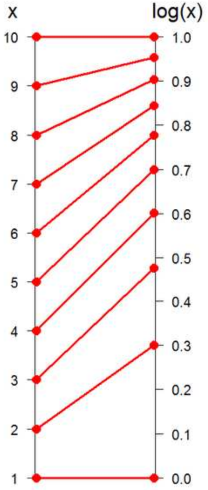


Logarithms and Benford's Law

$$
\begin{aligned}
&\text{Prob}(\text{leading digit } d) \\
&= \log_{10}(d+1) - \log_{10}(d) \\
&= \log_{10}\left(\frac{d+1}{d}\right) \\
&= \log_{10}\left(1 + \frac{1}{d}\right).
\end{aligned}
$$

Have Benford's law $\leftrightarrow$ mantissa of logarithms of data are uniformly distributed



The Power of the Right Perspective

Examples

- 2^n is Benford base 10 as $\log_{10} 2 \notin \mathbb{Q}$.

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- Most linear recurrence relations Benford:
  - ◇ $a_{n+1} = 2a_n - a_{n-1}$
  - ◇ take $a_0 = a_1 = 1$ or $a_0 = 0$, $a_1 = 1$.

Digits of 2^n

First 60 values of 2^n (only displaying 30)

			digit	#	Obs Prob	Benf Prob
1	1024	1048576	1	18	.300	.301
2	2048	2097152	2	12	.200	.176
4	4096	4194304	3	6	.100	.125
8	8192	8388608	4	6	.100	.097
16	16384	16777216	5	6	.100	.079
32	32768	33554432	6	4	.067	.067
64	65536	67108864	7	2	.033	.058
128	131072	134217728	8	5	.083	.051
256	262144	268435456	9	1	.017	.046
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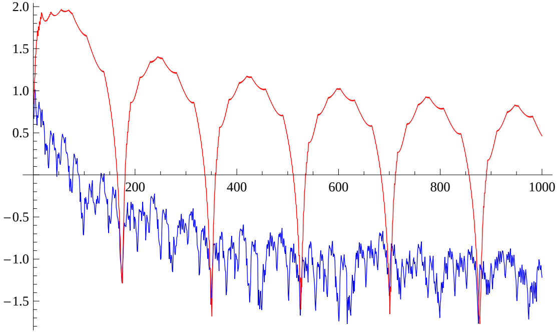
Logarithms and Benford's Law

χ^2 values for α^n, $1 \leq n \leq N$ (5% 15.5).

N	$\chi^2(\gamma)$	$\chi^2(e)$	$\chi^2(\pi)$
100	0.72	0.30	46.65
200	0.24	0.30	8.58
400	0.14	0.10	10.55
500	0.08	0.07	2.69
700	0.19	0.04	0.05
800	0.04	0.03	6.19
900	0.09	0.09	1.71
1000	0.02	0.06	2.90

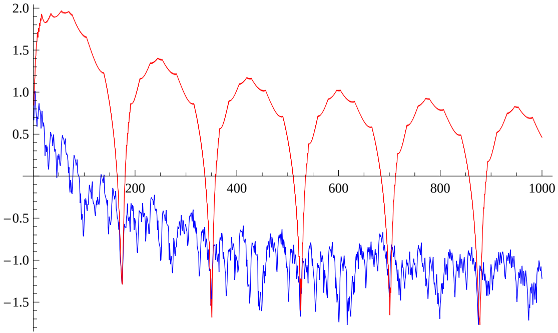
Logarithms and Benford's Law: Base 10 (5%: $\log(\chi^2) \approx 2.74$)

$\log(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1, \dots, N\}$.



Logarithms and Benford's Law: Base 10 (5%: $\log(\chi^2) \approx 2.74$)

$\log(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1, \dots, N\}$. Note $\pi^{175} \approx 1.0028 \cdot 10^{87}$.



New Result: Linear Recurrence Relations of Degree 2

- $a_{n+1} = f(n)a_n + g(n)a_{n-1}$ with non-constant coefficients $f(n)$ and $g(n)$.
- Explore conditions on f and g such that the sequence generated obeys Benford's Law for all initial values.
- First solve the closed form of the sequence (a_n), then analyze its main term.

- Main idea: reduce the degree of recurrence.
- $a_{n+1} = (\lambda(n) + \mu(n))a_n - \mu(n)\lambda(n-1)a_{n-1}$, and compare the coefficients:

$$f(n) = \lambda(n) + \mu(n)$$
$$g(n) = -\lambda(n-1)\mu(n).$$

- We show that for any given pair of f and g, such λ and μ always exist.

Linear Recurrence Relations of Degree 2

- Recurrence relations of degree 1:

$$\begin{aligned} a_{n+1} &= \lambda(n) a_n + b_n \\ b_n &= \mu(n) b_{n-1}. \end{aligned}$$

- $a_{n+1} = r(n)\left(1 + \sum_{k=3}^{n} \prod_{i=k}^{n} \frac{\lambda(i)}{\mu(i)} + \frac{a_2}{b_1} \prod_{i=2}^{n} \frac{\lambda(i)}{\mu(i)}\right),$

  where $r(n) := b_1 \prod_{i=2}^{n} \mu(i)$.

- Find conditions on μ, λ such that main term dominates; Benford if $\prod \mu(i)$ is.

Examples when f and g are functions

- If $\mu(k) = k$, then $r(n) = n!$.
- If $\mu(k) = k^\alpha$ where $\alpha \in \mathbb{R}$, then $r(n) = (n!)^\alpha$.
- If $\mu(k) = \exp(\alpha h(k))$ where α is irrational and $h(k)$ is a monic polynomial, then $\log r(n) = \alpha \sum_{k=1}^{n} h(k)$.

Lemma

The sequence $\{\alpha p(n)\}$ is equidistributed mod 1 if $\alpha \notin \mathbb{Q}$ and $p(n)$ a monic polynomial.

Examples when f and g are random variables

- Take $\mu(n) \sim h(n)U_n$ where the U_n's are independent uniform distributions on $[0,1]$, and $h(n)$ is a deterministic function in n such that $\prod_{i=1}^{n} h(i)$ is Benford.

  Then $r(n) = \prod_{i=1}^{n} h(i) \prod_{i=1}^{n} U_i$ is Benford.

- Take $\mu(n) \sim \exp(U_n)$ where the U_n's are i.i.d. random variables. Then take logarithm and sum up $\log(\mu(n))$. Apply Central Limit Theorem and get a Gaussian distribution

Linear Recurrences of Higher Degree

- Use recurrence relation of degree 3 as an example. Similar main idea: reduce the degree.

- Define the sequence $\{a_n\}_{n=1}^{\infty}$ by $a_{n+1} \ = \ f_1(n)a_n + f_2(n)a_{n-1} + f_3(n)a_{n-2}$.

- Define an auxiliary sequence $(b_n)_{n=1}^{\infty}$ by $b_n \ = \ a_{n+1} - \lambda(n)a_n$. Then (b_n) is degree 2.

Why Benford's Law?

Streets

Not all data sets satisfy Benford's Law.

- Long street $[1, L]$: $L = 199$ versus $L = 999$.
- Oscillates b/w $1/9$ and $5/9$ with first digit 1.

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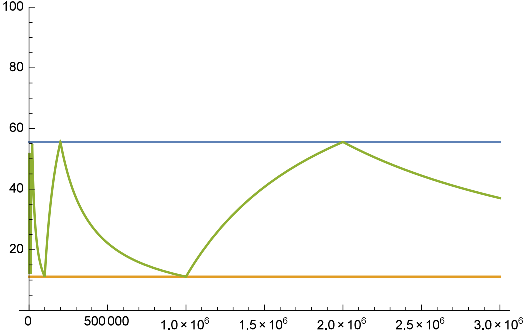

Probability first digit 1 versus street length L.

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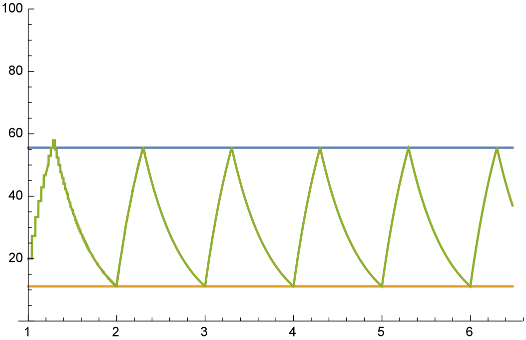

Probability first digit 1 versus log(street length L).

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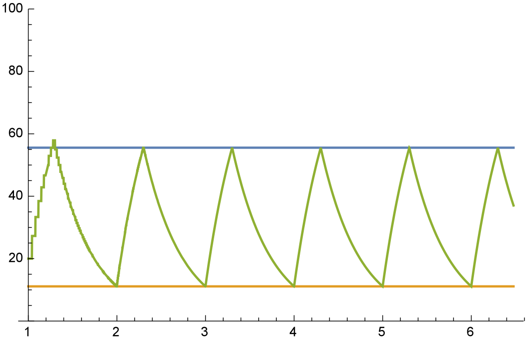
Not all data sets satisfy Benford's Law.

- Long street $[1, L]$: $L = 199$ versus $L = 999$.
- Oscillates b/w $1/9$ and $5/9$ with first digit 1.



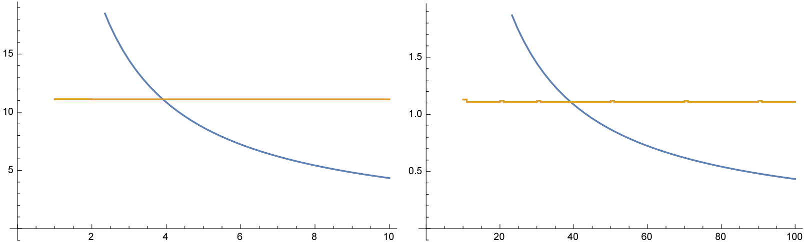
Probability first digit 1 versus log(street length L).

What if we have many streets of different lengths?

Amalgamating Streets

All houses: 1000 Streets,

each from 1 to 10000.



First digit and first two digits vs Benford.

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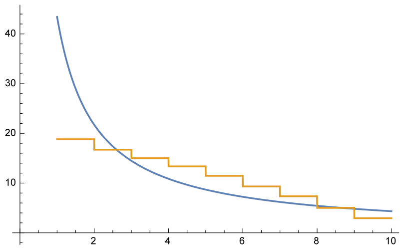

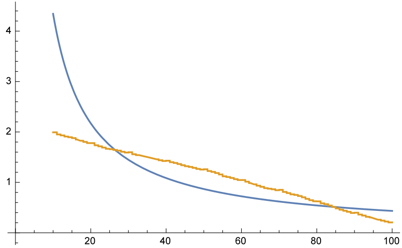

First digit and first two digits vs Benford.

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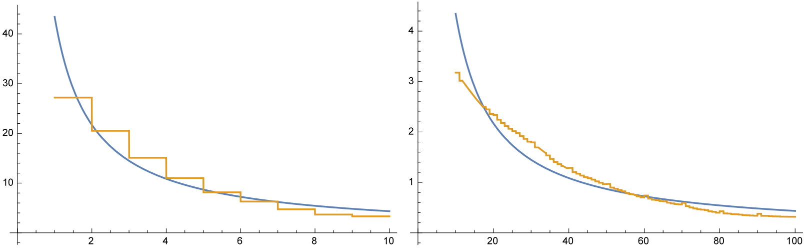

First digit and first two digits vs Benford.

Conclusion: More processes, closer to Benford.

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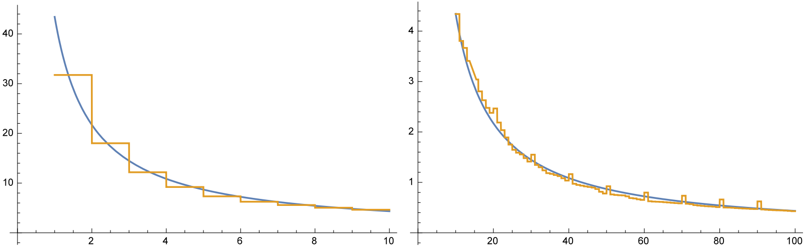
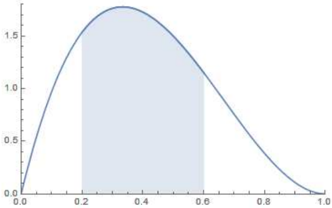


First digit and first two digits vs Benford.

Conclusion: More processes, closer to Benford.

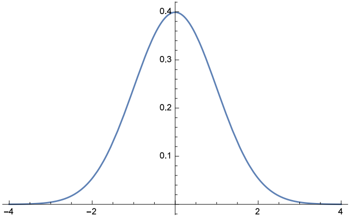
Probability Review



- Let X be random variable with density $p(x)$:
  - ◇ $p(x) \geq 0$; $\int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1$;
  - ◇ $\mathrm{Prob}\,(a \leq X \leq b) = \int_a^b p(x)\mathrm{d}x$.
- Mean $\mu = \int_{-\infty}^{\infty} x p(x)\mathrm{d}x$.
- Variance $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)\mathrm{d}x$.
- Independence: knowledge of one random variable gives no knowledge of the other.

Central Limit Theorem

$$\text{Normal } N(\mu, \sigma^2): \ p(x) \ = \ e^{-(x-\mu)^2/2\sigma^2}/\sqrt{2\pi\sigma^2}.$$



Theorem

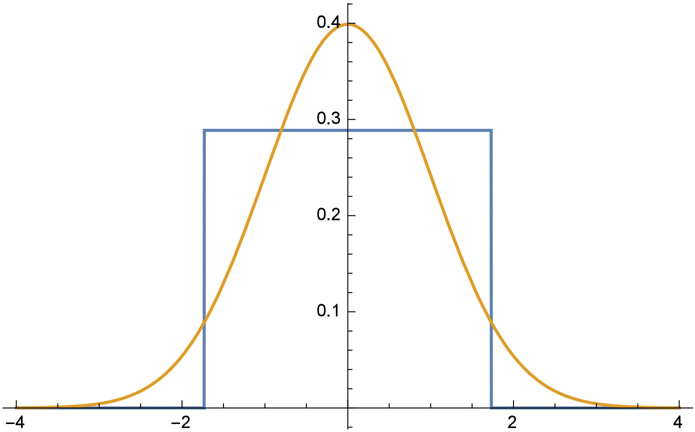
If $X_1, X_2, \dots$ independent, identically distributed random variables (mean μ, variance σ^2, finite moments) then

$$S_N \ := \ \frac{X_1 + \cdots + X_N - N\mu}{\sigma\sqrt{N}} \ \text{converges to} \ N(0,1).$$

Central Limit Theorem: Sums of Uniform Random Variables

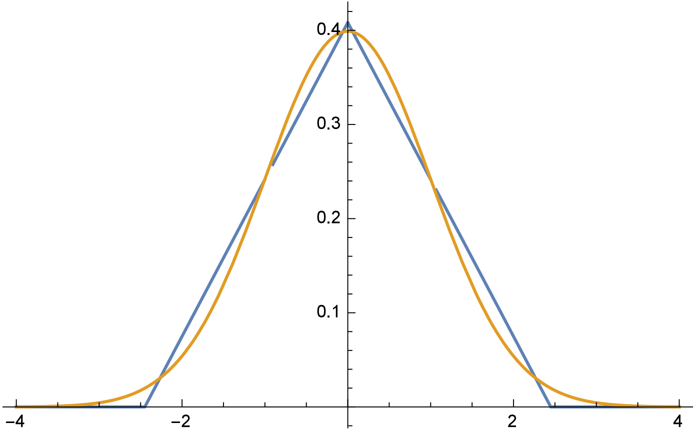
$X_i \sim \mathrm{Unif}(-1/2, 1/2)$ **(adjusted to mean 0, variance 1)**

$Y_1 = X_1/\sigma_{X_1}$ vs $N(0,1)$.



Central Limit Theorem: Sums of Uniform Random Variables
$X_i \sim \mathrm{Unif}(-1/2, 1/2)$ (adjusted to mean 0, variance 1)

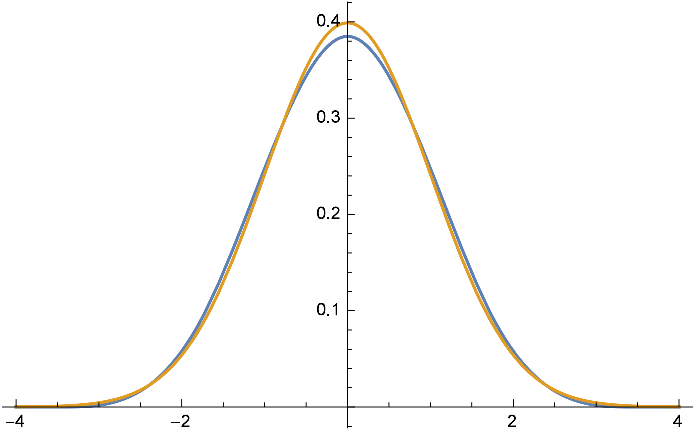
$$Y_2 = (X_1 + X_2)/\sigma_{X_1+X_2} \text{ vs } N(0,1).$$



Central Limit Theorem: Sums of Uniform Random Variables

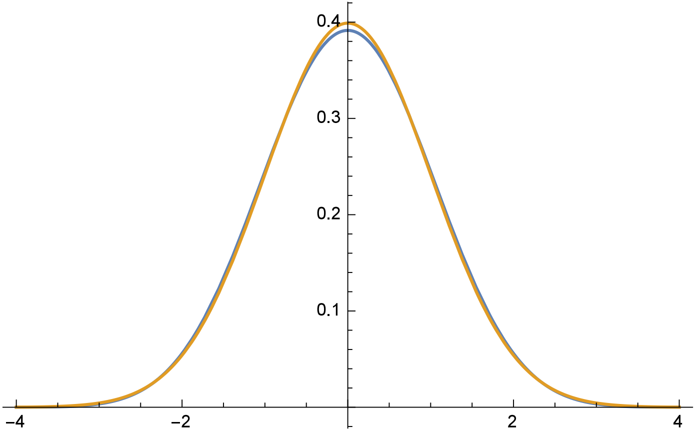
$X_i \sim \mathrm{Unif}(-1/2, 1/2)$ **(adjusted to mean 0, variance 1)**

$$Y_4 = (X_1 + X_2 + X_3 + X_4)/\sigma_{X_1+X_2+X_3+X_4} \text{ vs } N(0,1).$$



Central Limit Theorem: Sums of Uniform Random Variables $X_i \sim \mathrm{Unif}(-1/2, 1/2)$ (adjusted to mean 0, variance 1)

$$Y_8 = (X_1 + \cdots + X_8)/\sigma_{X_1+\cdots+X_8} \text{ vs } N(0,1).$$



Central Limit Theorem: Sums of Uniform Random Variables
$X_i \sim \mathrm{Unif}(-1/2, 1/2)$ (adjusted to mean 0, variance 1)

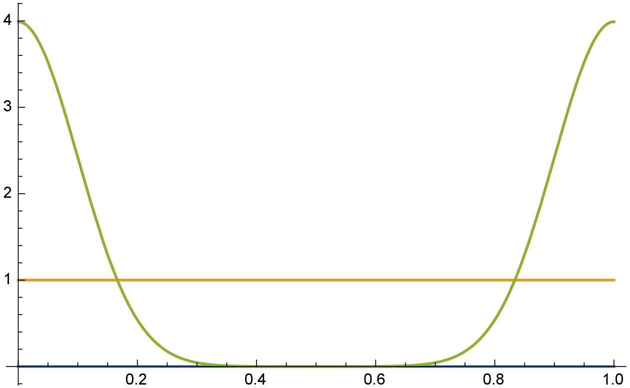
Density of $Y_4 = (X_1 + \cdots + X_4)/\sigma_{X_1+\cdots+X_4}$.

$$\frac{\begin{cases} \frac{1}{27}\left(18 + 9\sqrt{3}\, y - \sqrt{3}\, y^3\right) & y == 0 \\ \frac{1}{18}\left(12 - 6 y^2 - \sqrt{3}\, y^3\right) & -\sqrt{3} < y < 0 \\ \frac{1}{54}\left(72 - 36\sqrt{3}\, y + 18 y^2 - \sqrt{3}\, y^3\right) & \sqrt{3} < y < 2\sqrt{3} \\ \frac{1}{54}\left(18\sqrt{3}\, y - 18 y^2 + \sqrt{3}\, y^3\right) & y == \sqrt{3} \\ \frac{1}{18}\left(12 - 6 y^2 + \sqrt{3}\, y^3\right) & 0 < y < \sqrt{3} \\ \frac{1}{54}\left(72 + 36\sqrt{3}\, y + 18 y^2 + \sqrt{3}\, y^3\right) & -2\sqrt{3} < y \le -\sqrt{3} \\ 0 & \text{True} \end{cases}}{\sqrt{3}}$$

(Don't even think of asking to see Y_8's!)

Normal Distributions Mod 1

As $\sigma \to \infty$, $N(0, \sigma^2)$ mod 1 $\to$ Unif$(0, 1)$.



Variance is .01.

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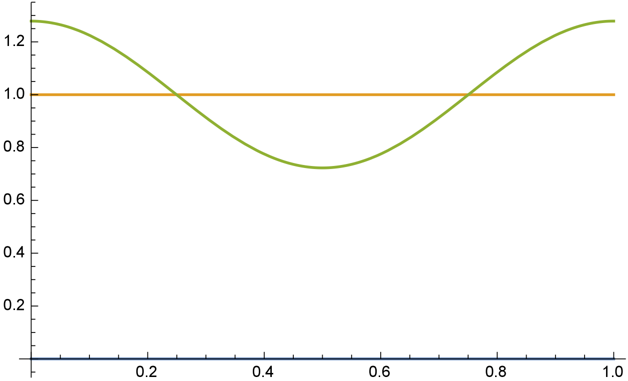

Variance is .1.

Normal Distributions Mod 1

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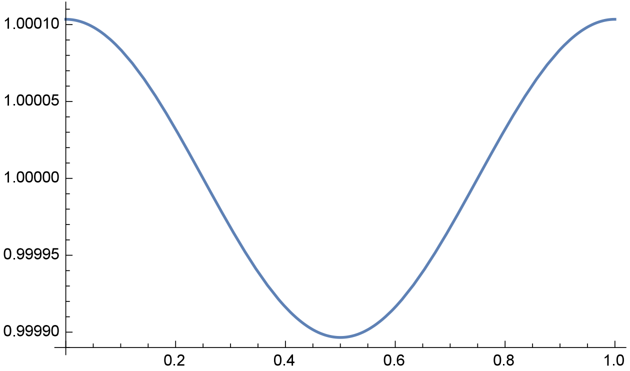

Variance is .5.

Products and Benford's Law

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$$V_N = \log_{10}(X_1 \cdot X_2 \cdots X_N)$$

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$$\begin{aligned} V_N &= \log_{10}(X_1 \cdot X_2 \cdots X_N) \\ &= \log_{10} X_1 + \log_{10} X_2 + \cdots + \log_{10} X_N \end{aligned}$$

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Need distribution of $V_N \bmod 1$, which by CLT becomes uniform, implying Benfordness!

Applications

Applications for the IRS: Detecting Fraud

A Tale of Two Steve Millers....

Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with 48 and 49, most due to one person.

- Write-off limit of $5,000. Officer had friends applying for credit cards, ran up balances just under $5,000 then he would write the debts off.

Can you see the cat in the tree?

Transmitting Images

How to transmit an image?

- Have an $L \times W$ grid with LW pixels.
- Each pixel a triple: (Red, Green, Blue).
- Often each value in $\{0, 1, 2, 3, \dots, 2^n - 1\}$.
- $n = 8$ gives 256 choices for each, or 16,777,216 possibilities.

Steganography

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Take one of the colors, say red, a number from 0 to 255.

Write in binary: $r_7 2^7 + r_6 2^6 + \cdots + r_1 2 + r_0$.

If change just the last or last two digits, very minor change to image.

Can you see the cat in the tree?

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Benford Good Processes

- A. Kontorovich and S. J. Miller, *Benford's Law, values of L-functions and the* $3x+1$ *problem*, Acta Arithmetica **120** (2005), no. 3, 269–297.

Poisson Summation and Benford's Law: Definitions

- Feller, Pinkham (often exact processes)

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- Poisson Summation Formula: f nice:

$$\sum_{\ell=-\infty}^{\infty} f(\ell) \ = \ \sum_{\ell=-\infty}^{\infty} \widehat{f}(\ell),$$

$$\text{Fourier transform } \widehat{f}(\xi) \ = \ \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx.$$

Benford Good Process

X_T is Benford Good if there is a nice f st

$$\mathrm{CDF}_{\vec{Y}_{T,B}}(y) \;=\; \int_{-\infty}^{y} \frac{1}{T} f\left(\frac{t}{T}\right) dt + E_T(y) \;:=\; G_T(y)$$

and monotonically increasing h ($h(|T|) \to \infty$):

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- Decay of the Fourier Transform: $\sum_{\ell \neq 0} \left| \frac{\widehat{f}(T\ell)}{\ell} \right| = o(1)$.
- Small translated error: $\mathcal{E}(a, b, T)) = \sum_{|\ell| \leq Th(T)} [E_T(b + \ell) - E_T(a + \ell)] = o(1)$.

Main Theorem

Theorem (Kontorovich and M–, 2005)

X_T converging to X as $T \to \infty$ (think spreading Gaussian). If X_T is Benford good, then X is Benford.

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- Examples
  - ◇ L-functions
  - ◇ characteristic polynomials (RMT)
  - ◇ $3x + 1$ problem
  - ◇ geometric Brownian motion.

Sketch of the proof

- Structure Theorem:
  - ◇ main term is something nice spreading out
  - ◇ apply Poisson summation

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  - ◇ apply Poisson summation
- Control translated errors:
  - ◇ hardest step
  - ◇ techniques problem specific

Sketch of the proof (continued)

$$\sum_{\ell=-\infty}^{\infty} \mathbb{P}\left(a + \ell \leq \overrightarrow{Y}_{T,B} \leq b + \ell\right)$$

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Sketch of the proof (continued)

$$\begin{aligned}
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= &\int_a^b \sum_{|\ell| \leq Th(T)} \frac{1}{T} f\left(\frac{t+\ell}{T}\right) dt + \mathcal{E}(a,b,T) + o(1)
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$$= \widehat{f}(0)\cdot(b-a) + \sum_{\ell \ne 0} \widehat{f}(T\ell) \frac{e^{2\pi i b\ell} - e^{2\pi i a\ell}}{2\pi i \ell} + o(1).$$

Riemann Zeta Function (for real part of s greater than 1)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

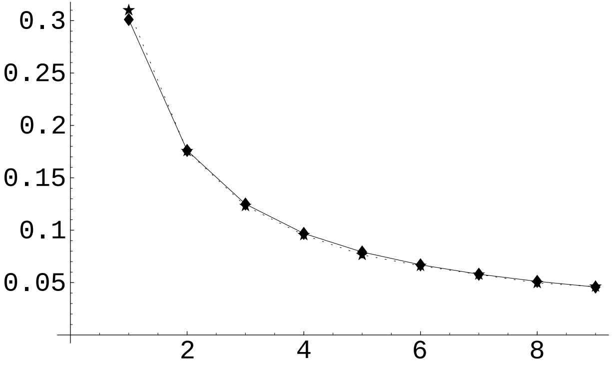
Geometric Series Formula: $(1-x)^{-1} = 1 + x + x^2 + \cdots$.
Unique Factorization: $n = p_1^{r_1} \cdots p_m^{r_m}$.

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \left[1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 + \cdots\right]\left[1 + \frac{1}{3^s} + \left(\frac{1}{3^s}\right)^2 + \cdots\right]\cdots$$

$$= \sum_n \frac{1}{n^s}.$$

Riemann Zeta Function

$$\left|\zeta\left(\frac{1}{2} + i\frac{k}{4}\right)\right|, k \in \{0, 1, \ldots, 65535\}.$$



$3x + 1$ Problem

- Kakutani (conspiracy), Erdös (not ready).
- x odd, $T(x) = \frac{3x+1}{2^k}$, $2^k || 3x + 1$.
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  2-path $(1, 1)$, 5-path $(1, 1, 2, 3, 4)$.
  m-path: $(k_1, \dots, k_m)$.

$3x+1$ and Benford

Theorem (Kontorovich and M–, 2005)

As $m \to \infty$, $x_m/(3/4)^m x_0$ is Benford.

Theorem (Lagarias-Soundararajan, 2006)

$X \geq 2^N$, for all but at most $c(B)N^{-1/36}X$ initial seeds the distribution of the first N iterates of the $3x+1$ map are within $2N^{-1/36}$ of the Benford probabilities.

$3x+1$ Data: random 10,000 digit number, $2^k||3x+1$

80,514 iterations ($(4/3)^n = a_0$ predicts 80,319); $\chi^2 = 13.5$ (5% 15.5).

Digit	Number	Observed	Benford
1	24251	0.301	0.301
2	14156	0.176	0.176
3	10227	0.127	0.125
4	7931	0.099	0.097
5	6359	0.079	0.079
6	5372	0.067	0.067
7	4476	0.056	0.058
8	4092	0.051	0.051
9	3650	0.045	0.046

$3x+1$ Data: random 10,000 digit number, $2|3x+1$

241,344 iterations, $\chi^2 = 11.4$ (5% 15.5).

Digit	Number	Observed	Benford
1	72924	0.302	0.301
2	42357	0.176	0.176
3	30201	0.125	0.125
4	23507	0.097	0.097
5	18928	0.078	0.079
6	16296	0.068	0.067
7	13702	0.057	0.058
8	12356	0.051	0.051
9	11073	0.046	0.046

Stick Decomposition

- T. Becker, D. Burt, T. C. Corcoran, A. Greaves-Tunnell, J. R. Iafrate, J. Jing, S. J. Miller, J. D. Porfilio, R. Ronan, J. Samranvedhya, F. W. Strauch and B. Talbut, *Benford's Law and Continuous Dependent Random Variables*, Annals of Physics **388** (2018), 350–381.
- J. Iafrate, S. J. Miller and F. W. Strauch, *Equipartitions and a distribution for numbers: A statistical model for Benford's law*, Physical Review E **91** (2015), no. 6, 062138 (6 pages).

Fixed Proportion Decomposition Process

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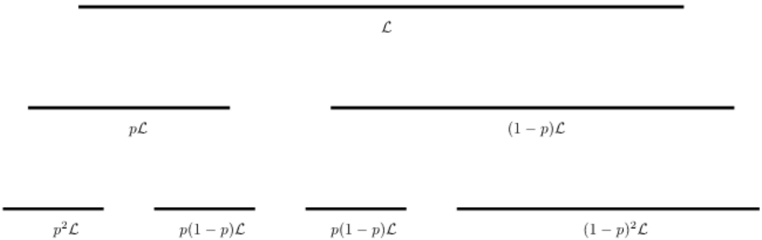
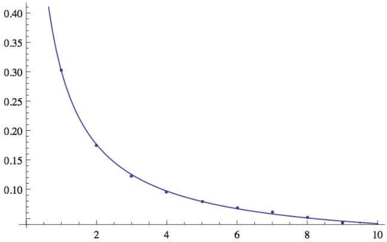
2. Uniformly choose a proportion $p \in (0, 1)$.
3. Break the stick into two pieces—lengths $p\mathcal{L}$ and $(1 - p)\mathcal{L}$.
4. Repeat N times (using the same proportion).

Fixed Proportion Decomposition Process

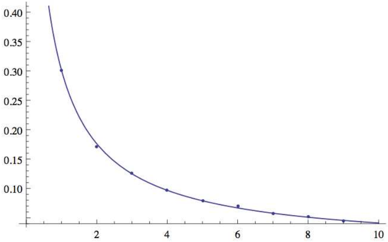


Fixed Proportion Conjecture (Joy Jing '13)

Conjecture: The above decomposition process is Benford as $N \to \infty$ for any $p \in (0,1)$, $p \neq \frac{1}{2}$.



(B) $p = 0.51$ and $N = 10000$.



(B) $p = 0.99$ and $N = 50000$. Benford distribution overlaid.

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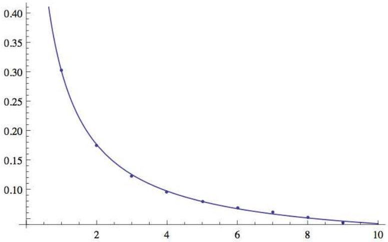
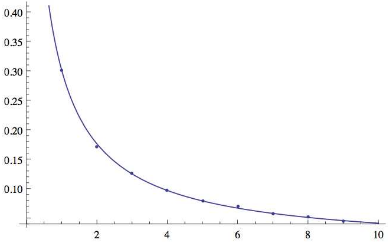


(B) $p = 0.51$ and $N = 10000$.



(B) $p = 0.99$ and $N = 50000$. Benford distribution overlaid.

Counterexample (SMALL REU '13): $p = \frac{1}{11}$, $1 - p = \frac{10}{11}$.

Benford Analysis

At N^{th} level,

- 2^N sticks
- $N+1$ distinct lengths: write $p^{N-j}(1-p)^j$ as

$$p^N\left(\frac{1-p}{p}\right)^j,\ \ j \in \{0,\dots,N\},\ \text{have}\ \binom{N}{j}\ \text{times.}$$

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(Weighted) Geometric with ratio $\frac{1-p}{p} = 10^y$;
behavior depends on irrationality of y!
Theorem: Benford if and only if y irrational.

Benford Analysis (cont)

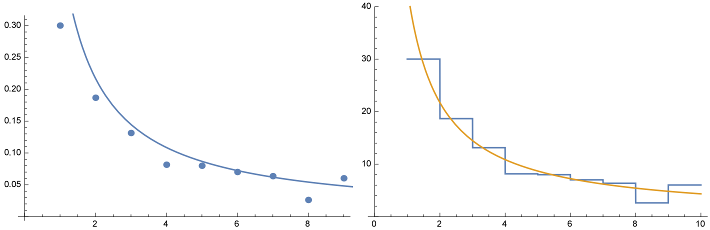
Say $\frac{1-p}{p} = 10^{r/q}$ for r, q integers.

All terms with index j mod q have same leading digit; probability index $j \mod q$ is

$$\begin{aligned}
\frac{1}{2^N}\left[\binom{N}{j}+\binom{N}{j+q}+\binom{N}{j+2q}+\cdots\right] &= \frac{1}{q}\sum_{s=0}^{q-1}\left(\cos\frac{\pi s}{q}\right)^N \cos\frac{\pi(N-2j)s}{q} \\
&= \frac{1}{q}\left(1+\sum_{s=1}^{q-1}\left(\cos\frac{\pi s}{q}\right)^N \cos\frac{\pi(N-2j)s}{q}\right) \\
&= \frac{1}{q}\left(1+\mathrm{Err}\left[(q-1)\left(\cos\frac{\pi}{q}\right)^N\right]\right),
\end{aligned}$$

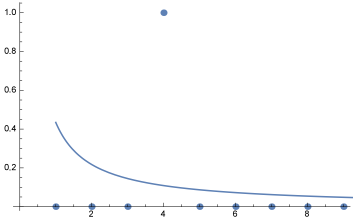
where $\mathrm{Err}[X]$ indicates an absolute error of size at most X

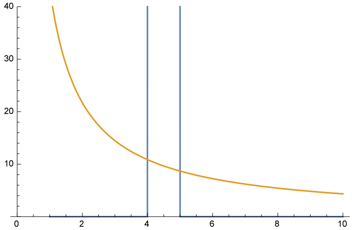
Examples



$p = 3/11$, 1000 levels; $y = \log_{10}(8/3) \notin \mathbb{Q}$ (irrational)

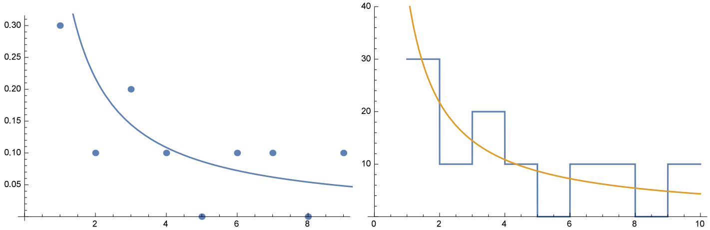
Examples





$p = 1/11$, 1000 levels; $y = 1 \in \mathbb{Q}$ (rational)

Examples



$p = 1/(1 + 10^{33/10})$, 1000 levels; $y = 33/10 \in \mathbb{Q}$ (rational)

Random Cuts

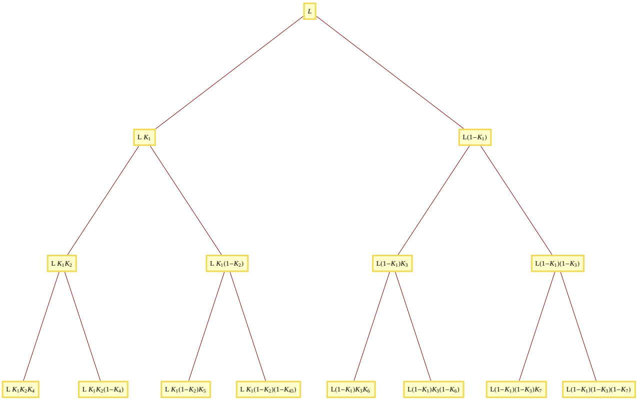


Figure: Unrestricted Decomposition: Breaking L into pieces, $N = 3$.

Conclusions and References

Conclusions and Future Investigations

- See many different systems exhibit Benford behavior.
- Ingredients of proofs (logarithms, equidistribution).
- Applications to fraud detection / data integrity.

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Products of Random Variables

- S. J. Miller and M. Nigrini, *The Modulo 1 Central Limit Theorem and Benford's Law for Products*, International Journal of Algebra **2** (2008), no. 3, 119–130.

Preliminaries

- $X_1 \cdots X_n \Leftrightarrow Y_1 + \cdots + Y_n \bmod 1$, $Y_i = \log_B X_i$

- Density Y_i is g_i, density $Y_i + Y_j$ is

$$(g_i * g_j)(y) \ = \ \int_0^1 g_i(t) g_j(y-t) dt.$$

- $h_n = g_1 * \cdots * g_n$, $\widehat{h}_n(\xi) = \widehat{g}_1(\xi) \cdots \widehat{g}_n(\xi)$.

Modulo 1 Central Limit Theorem

Theorem (M– and Nigrini 2007)

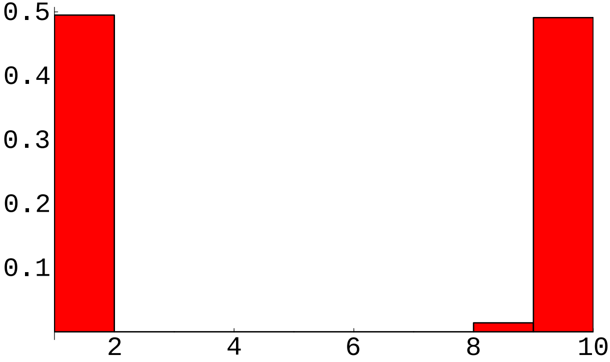
$\{Y_m\}$ *independent continuous random variables on* $[0,1)$ *(not necc. i.i.d.), densities* $\{g_m\}$*.* $Y_1 + \cdots + Y_M \bmod 1$ *converges to the uniform distribution as* $M \to \infty$ *in* $L^1([0,1])$ *if and only if for all* $n \neq 0$, $\lim_{M\to\infty} \widehat{g}_1(n) \cdots \widehat{g_M}(n) = 0$*.*

◇ Gives info on rate of convergence.

Generalizations

- Levy proved for i.i.d.r.v. just one year after Benford's paper.
- Generalized to other compact groups, with estimates on the rate of convergence.
  ◇ Stromberg: n-fold convolution of a regular probability measure on a compact Hausdorff group G converges to normalized Haar measure in weak-star topology iff support of the distribution not contained in a coset of a proper normal closed subgroup of G.

Distribution of digits (base 10) of 1000 products $X_1 \cdots X_{1000}$, where $g_{10,m} = \phi_{11^m}$.
$\phi_m(x) = m$ if $|x - 1/8| \le 1/2m$ (0 otherwise).



Proof under stronger conditions

- Use standard CLT to show $Y_1 + \cdots + Y_M$ tends to a Gaussian.
- Use Poisson Summation to show the Gaussian tends to the uniform modulo 1.

Proof under stronger conditions

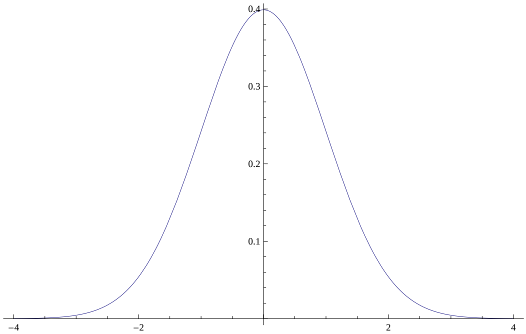


Figure: Plot of normal (mean 0, stdev 1).

Proof under stronger conditions

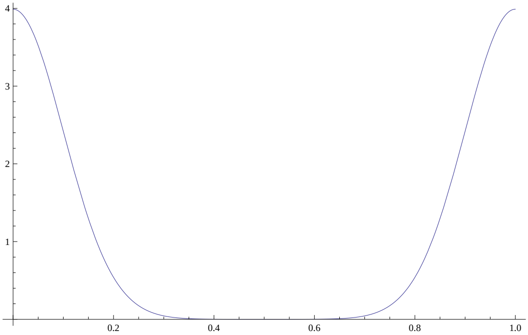


Figure: Plot of normal (mean 0, stdev .1) modulo 1.

Proof under stronger conditions

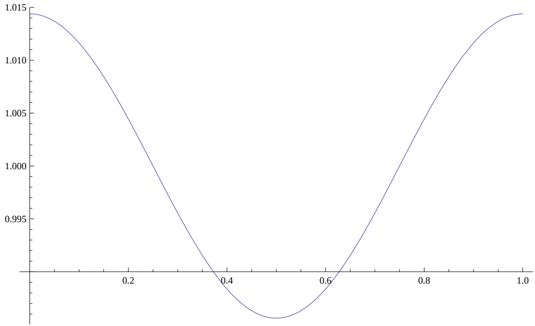


Figure: Plot of normal (mean 0, stdev .5) modulo 1.

Inputs

Poisson Summation Formula

f nice:

$$\sum_{\ell=-\infty}^{\infty} f(\ell) \ = \ \sum_{\ell=-\infty}^{\infty} \widehat{f}(\ell),$$

$$\text{Fourier transform } \widehat{f}(\xi) \ = \ \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx.$$

Lemma

$$\frac{2}{\sqrt{2\pi\sigma^2}} \int_{\sigma^{1+\delta}}^{\infty} e^{-x^2/2\sigma^2} dx \ \ll \ e^{-\sigma^{2\delta}/2}.$$

Proof Under Weaker Conditions

Lemma

As $N \to \infty$, $p_N(x) = \frac{e^{-\pi x^2/N}}{\sqrt{N}}$ becomes equidistributed modulo 1.

- $\int_{\substack{x=-\infty \\ x \bmod 1 \in [a,b]}}^{\infty} p_N(x)dx = \frac{1}{\sqrt{N}} \sum_{n\in\mathbb{Z}} \int_{x=a}^{b} e^{-\pi(x+n)^2/N} dx.$
- $e^{-\pi(x+n)^2/N} = e^{-\pi n^2/N} + O\left(\frac{\max(1,|n|)}{N} e^{-n^2/N}\right).$
- Can restrict sum to $|n| \le N^{5/4}$.
- $\frac{1}{\sqrt{N}} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/N} = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 N}.$

Proof Under Weaker Conditions

$$\begin{aligned}
&\frac{1}{\sqrt{N}} \sum_{|n| \le N^{5/4}} \int_{x=a}^{b} e^{-\pi (x+n)^2/N} dx \\
&= \frac{1}{\sqrt{N}} \sum_{|n| \le N^{5/4}} \int_{x=a}^{b} \left[e^{-\pi n^2/N} + O\left(\frac{\max(1,|n|)}{N} e^{-n^2/N} \right) \right] dx \\
&= \frac{b-a}{\sqrt{N}} \sum_{|n| \le N^{5/4}} e^{-\pi n^2/N} + O\left(\frac{1}{N} \sum_{n=0}^{N^{5/4}} \frac{n+1}{\sqrt{N}} e^{-\pi (n/\sqrt{N})^2} \right) \\
&= \frac{b-a}{\sqrt{N}} \sum_{|n| \le N^{5/4}} e^{-\pi n^2/N} + O\left(\frac{1}{N} \int_{w=0}^{N^{3/4}} (w+1) e^{-\pi w^2} \sqrt{N} dw \right) \\
&= \frac{b-a}{\sqrt{N}} \sum_{|n| \le N^{5/4}} e^{-\pi n^2/N} + O\left(N^{-1/2} \right).
\end{aligned}$$

Proof Under Weaker Conditions

Extend sums to $n \in \mathbb{Z}$, apply Poisson Summation:

$$\frac{1}{\sqrt{N}} \sum_{n \in \mathbb{Z}} \int_{x=a}^{b} e^{-\pi (x+n)^2/N} dx \;\approx\; (b-a) \cdot \sum_{n \in \mathbb{Z}} e^{-\pi n^2 N}.$$

For $n = 0$ the right hand side is $b - a$.

For all other n, we trivially estimate the sum:

$$\sum_{n \neq 0} e^{-\pi n^2 N} \;\leq\; 2 \sum_{n \geq 1} e^{-\pi n N} \;\leq\; \frac{2e^{-\pi N}}{1 - e^{-\pi N}},$$

which is less than $4e^{-\pi N}$ for N sufficiently large.

Proof in General Case: Fourier input

- Fejér kernel:

$$F_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x}.$$

- Fejér series $T_N f(x)$ equals

$$(f * F_N)(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \widehat{f}(n) e^{2\pi i n x}.$$

- Lebesgue's Theorem: $f \in L^1([0,1])$. As $N \to \infty$, $T_N f$ converges to f in $L^1([0,1])$.
- $T_N(f * g) = (T_N f) * g$: convolution assoc.

Proof of Modulo 1 CLT

- Density of sum is $h_\ell = g_1 * \cdots * g_\ell$.
- Suffices show $\forall \epsilon$: $\lim_{M\to\infty} \int_0^1 |h_M(x) - 1| dx < \epsilon$.
- Lebesgue's Theorem: N large,

$$||h_1 - T_N h_1||_1 = \int_0^1 |h_1(x) - T_N h_1(x)| dx < \frac{\epsilon}{2}.$$

- Claim: above holds for h_M for all M.

Proof of Modulo 1 CLT : Proof of Claim

$$T_N h_{M+1} = T_N(h_M * g_{M+1}) = (T_N h_M) * g_{M+1}$$

$$\begin{aligned}
||h_{M+1} - T_N h_{M+1}||_1 &= \int_0^1 |h_{M+1}(x) - T_N h_{M+1}(x)| dx \\
&= \int_0^1 |(h_M * g_{M+1})(x) - (T_N h_M) * g_{M+1}(x)| dx \\
&= \int_0^1 \left| \int_0^1 (h_M(y) - T_N h_M(y))\, g_{M+1}(x-y) \right| dydx \\
&\leq \int_0^1 \int_0^1 |h_M(y) - T_N h_M(y)| g_{M+1}(x-y) dxdy \\
&= \int_0^1 |h_M(y) - T_N h_M(y)| dy \cdot 1 \ < \ \frac{\epsilon}{2}.
\end{aligned}$$

Proof of Modulo 1 CLT

Show $\lim_{M\to\infty} ||h_M - 1||_1 = 0$.
Triangle inequality:

$$||h_M - 1||_1 \le ||h_M - T_N h_M||_1 + ||T_N h_M - 1||_1.$$

Choices of N and ϵ:

$$||h_M - T_N h_M||_1 < \epsilon/2.$$

Show $||T_N h_M - 1||_1 < \epsilon/2$.

Proof of Modulo 1 CLT

$$||T_N h_M - 1||_1 = \int_0^1 \left| \sum_{\substack{n=-N \\ n\neq 0}}^{N} \left(1 - \frac{|n|}{N}\right) \widehat{h_M}(n) e^{2\pi i n x} \right| dx$$

$$\leq \sum_{\substack{n=-N \\ n\neq 0}}^{N} \left(1 - \frac{|n|}{N}\right) |\widehat{h_M}(n)|$$

$\widehat{h_M}(n) = \widehat{g_1}(n) \cdots \widehat{g_M}(n) \longrightarrow_{M\to\infty} 0.$
For fixed N and ϵ, choose M large so that $|\widehat{h_M}(n)| < \epsilon/4N$ whenever $n \neq 0$ and $|n| \leq N$.

Products and Chains of Random Variables

- D. Jang, J. U. Kang, A. Kruckman, J. Kudo and S. J. Miller, *Chains of distributions, hierarchical Bayesian models and Benford's Law*, Journal of Algebra, Number Theory: Advances and Applications, volume 1, number 1 (March 2009), 37–60.

Key Ingredients

- Mellin transform and Fourier transform related by logarithmic change of variable.
- Poisson summation from collapsing to modulo 1 random variables.

Preliminaries

- $\Xi_1, \dots, \Xi_n$ nice independent r.v.'s on $[0, \infty)$.
- Density $\Xi_1 \cdot \Xi_2$:

$$\int_0^\infty f_2\left(\frac{x}{t}\right) f_1(t) \frac{dt}{t}$$

Preliminaries

- $\Xi_1, \dots, \Xi_n$ nice independent r.v.'s on $[0, \infty)$.
- Density $\Xi_1 \cdot \Xi_2$:

$$\int_0^\infty f_2\left(\frac{x}{t}\right) f_1(t) \frac{dt}{t}$$

$\diamond$ Proof: $\text{Prob}(\Xi_1 \cdot \Xi_2 \in [0, x])$:

$$\int_{t=0}^\infty \text{Prob}\left(\Xi_2 \in \left[0, \frac{x}{t}\right]\right) f_1(t) dt$$

$$= \int_{t=0}^\infty F_2\left(\frac{x}{t}\right) f_1(t) dt,$$

differentiate.

Mellin Transform

$$(\mathcal{M}f)(s) = \int_0^\infty f(x)x^s \frac{dx}{x}$$

$$(\mathcal{M}^{-1}g)(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(s)x^{-s}ds$$

$$g(s) = (\mathcal{M}f)(s), f(x) = (\mathcal{M}^{-1}g)(x).$$

$$(f_1 \star f_2)(x) = \int_0^\infty f_2\left(\frac{x}{t}\right) f_1(t)\frac{dt}{t}$$

$$(\mathcal{M}(f_1 \star f_2))(s) = (\mathcal{M}f_1)(s) \cdot (\mathcal{M}f_2)(s).$$

Mellin Transform Formulation: Products Random Variables

Theorem

X_i's independent, densities f_i. $\Xi_n = X_1 \cdots X_n$,

$$\begin{aligned} h_n(x_n) &= (f_1 \star \cdots \star f_n)(x_n) \\ (\mathcal{M}h_n)(s) &= \prod_{m=1}^{n} (\mathcal{M}f_m)(s). \end{aligned}$$

As $n \to \infty$, Ξ_n becomes Benford: $Y_n = \log_B \Xi_n$,
$|\text{Prob}(Y_n \bmod 1 \in [a,b]) - (b-a)| \le$

$$(b-a) \cdot \sum_{\ell \neq 0, \ell=-\infty}^{\infty} \prod_{m=1}^{n} (\mathcal{M}f_i) \left(1 - \frac{2\pi i \ell}{\log B}\right).$$

Proof of Kossovsky's Chain Conjecture for certain densities

Conditions

- $\{\mathcal{D}_i(\theta)\}_{i\in I}$: one-parameter distributions, densities $f_{\mathcal{D}_i(\theta)}$ on $[0,\infty)$.
- $p : \mathbb{N} \to I$, $X_1 \sim \mathcal{D}_{p(1)}(1)$, $X_m \sim \mathcal{D}_{p(m)}(X_{m-1})$.
- $m \geq 2$,

$$f_m(x_m) \ = \ \int_0^\infty f_{\mathcal{D}_{p(m)}(1)}\left(\frac{x_m}{x_{m-1}}\right) f_{m-1}(x_{m-1})\frac{dx_{m-1}}{x_{m-1}}$$

- $$\lim_{n\to\infty} \sum_{\substack{\ell=-\infty \\ \ell\neq 0}}^{\infty} \prod_{m=1}^{n} (\mathcal{M}f_{\mathcal{D}_{p(m)}(1)})\left(1 - \frac{2\pi i\ell}{\log B}\right) \ = \ 0$$

Chains of Random Variables

Return to street problem: chain of uniforms.

Let $\mathcal{D}_{\rm unif}(\theta)$ be the density of a uniform random variable on $[0, \theta]$.

Let $X_1 \sim \mathcal{D}_{\rm unif}(1)$ and $X_{n+1} \sim \mathcal{D}_{\rm unif}(X_n)$.

Proof of Kossovsky's Chain Conjecture for certain densities

Theorem (JKKKM)

- *If conditions hold, as $n \to \infty$ the distribution of leading digits of X_n tends to Benford's law.*
- *The error is a nice function of the Mellin transforms: if $Y_n = \log_B X_n$, then*

$$\left|\mathrm{Prob}(Y_n \bmod 1 \in [a,b]) - (b+a)\right| \leq$$

$$\left|(b-a) \cdot \sum_{\substack{\ell=-\infty \\ \ell \neq 0}}^{\infty} \prod_{m=1}^{n} \left(\mathcal{M} f_{\mathcal{D}_{p(m)}}(1)\right) \left(1 - \frac{2\pi i \ell}{\log B}\right)\right|$$

Example: All $X_i \sim \mathrm{Exp}(1)$

- $X_i \sim \mathrm{Exp}(1)$, $Y_n = \log_B \Xi_n$.
- Needed ingredients:
  - ◇ $\int_0^\infty \exp(-x) x^{s-1} dx = \Gamma(s)$.
  - ◇ $|\Gamma(1+ix)| = \sqrt{\pi x / \sinh(\pi x)}, x \in \mathbb{R}$.
- $|P_n(s) - \log_{10}(s)| \le$

$$\log_B s \sum_{\ell=1}^{\infty} \left(\frac{2\pi^2 \ell / \log B}{\sinh(2\pi^2 \ell / \log B)} \right)^{n/2}.$$

Example: All $X_i \sim \mathrm{Exp}(1)$

Bounds on the error

- $|P_n(s) - \log_{10} s| \le$
  - $3.3 \cdot 10^{-3} \log_B s \quad \text{if } n = 2,$
  - $1.9 \cdot 10^{-4} \log_B s \quad \text{if } n = 3,$
  - $1.1 \cdot 10^{-5} \log_B s \quad \text{if } n = 5, \text{ and}$
  - $3.6 \cdot 10^{-13} \log_B s \text{ if } n = 10.$
- Error at most

$$\log_{10} s \sum_{\ell=1}^{\infty} \left(\frac{17.148\ell}{\exp(8.5726\ell)} \right)^{n/2} \le .057^n \log_{10} s$$