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d'Andria, Diego

DeBacker, Jason

Evans, Richard W.

Pycroft, Jonathan

van der Wielen, Wouter

Zachłod-Jelec, Magdalena

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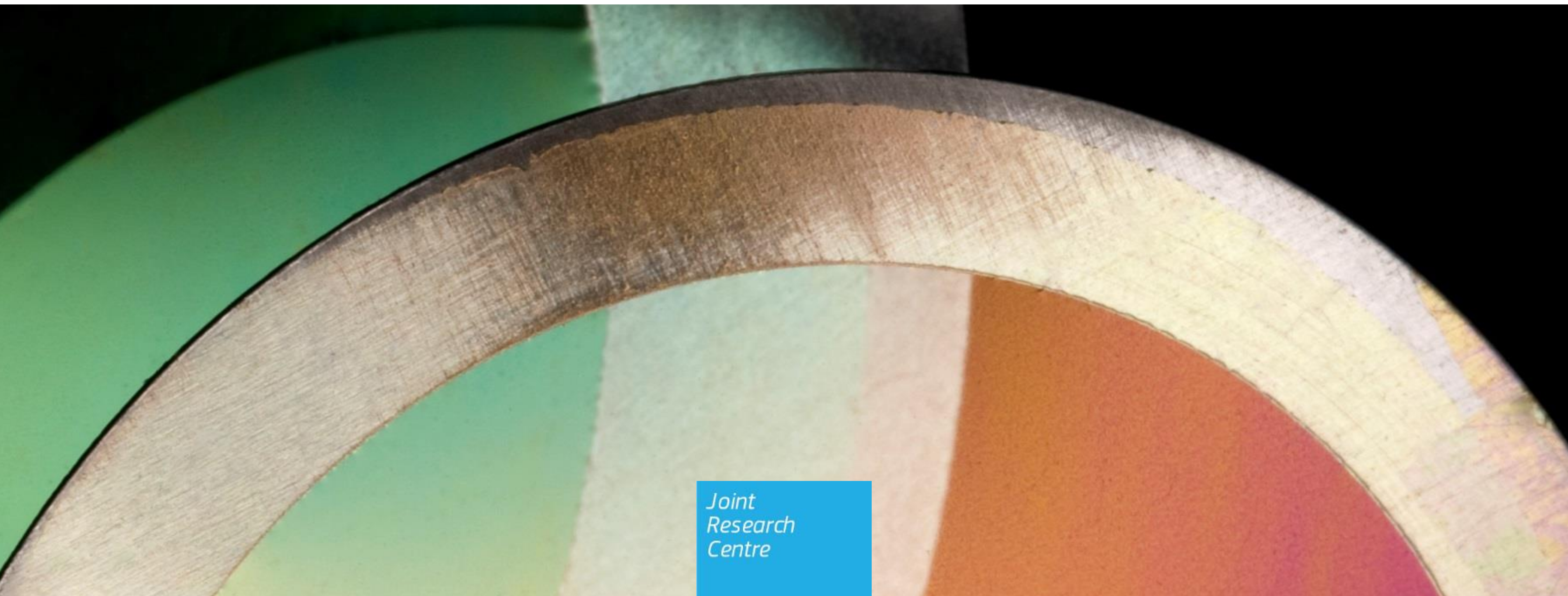

Joint Research Centre

Contact information
Name: Magdalena Zachlod-Jelec
Email: magdalena.zachlod-jelec@ec.europa.eu

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EDGE-M3: A Dynamic General Equilibrium Micro-Macro Model for the EU Member States

Diego d'Andria *[a], Jason DeBacker †[b], Richard W. Evans ‡[c], Jonathan Pycroft §[a], Wouter van der Wielen ¶[a] and Magdalena Zachłod-Jelec ‖[a]

[a]European Commission Joint Research Centre, Seville, Spain
[b]Darla Moore School of Business, University of South Carolina
[c]Computational Social Science, University of Chicago

Abstract

This paper provides a technical description of the overlapping generations model used by the Joint Research Centre to analyse tax policy reforms, including in particular pension and demographic issues. The main feature of the `EDGE-M3` model lies in its high level of disaggregation and the close connection between microeconomic and macroeconomic mechanisms which makes it a very suitable model to analyse the redistributive impact of policies. `EDGE-M3` features eighty generations and seven earnings-ability types of individuals. To facilitate a realistic dynamic population structure `EDGE-M3` includes Eurostat's demographic projections. In terms of calibration, the `EDGE-M3` family of overlapping generations models is heavily calibrated on microeconomic data. This allows the introduction of the underlying individuals' characteristics in a macro model to the greatest extent possible. In particular, it includes the richness of the tax code by means of income tax and social insurance contribution rate functions estimated using data from the `EUROMOD` microsimulation model. This feature allows in particular a close connection between the macro and the micro model. In addition, the earnings profiles of the seven heterogeneous agent types are estimated using survey data. Finally, the labour supply, bequests and consumption tax calibration are all done using detailed microeconomic data, making the model highly suitable for the analysis of intra- and intergenerational analysis of tax policy.

JEL classification: H24, H31, D15, D58, E62, J22

Keywords: computable general equilibrium, overlapping generations, heterogeneous ability, fiscal policy, microsimulation

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The views expressed are those of the authors and may not in any circumstances be regarded as stating an official position of the European Commission.

*Email: diego.d'andria@ec.europa.eu
†Email: jason.debacker@moore.sc.edu
‡Email: rwevans@uchicago.edu
§Email: jonathan.pycroft@outlook.com
¶Email: wouter.van-der-wielen@ec.europa.eu
‖Email: magdalena.zachlod-jelec@ec.europa.eu, corresponding author

Contents

1 Introduction

In this paper we present EDGE-M3, an overlapping generations (OLG) model for the EU countries.[1] We discuss the model structure in detail, presenting all the equations of the model, and take the example of Italy to illustrate its calibration.

The most notable characteristic that differentiates an overlapping generations model from other dynamic general equilibrium models is its more realistic representation of the finite lifetimes of individuals and the cross-sectional age heterogeneity that exists in the economy. One can make a strong case that age heterogeneity and income heterogeneity are two of the main sources of diversity that explain much of the behaviour in which analysts and policy-makers are interested, for instance when studying the differential effects of tax policy burdening different categories of taxpayers. The model not only provides a steady-state solution, but also simulates the transition of the economy from the initial state to the steady-state, which provides the analyst with useful insight about the timing of economic responses to a policy reform.

EDGE-M3 is a general equilibrium model, which implies that the behaviour of households and firms can cause macroeconomic variables and prices to adjust. EDGE-M3 is dynamic, in the sense that households in the model make consumption, savings, and labour supply decisions based on expectations over their entire lifetime, not just the current period. These choices, in turn, dynamically affect the aggregate stock of capital thus affecting future production. The model features overlapping generations of households endowed with different levels of productivity (which we also refer to as "ability types" in the text), meaning that within each simulated period, households of different age and income coexist. Each household in the model decides how much to participate to the labour force and how to allocate earned income between consumption and savings, knowing that any residual wealth at the end of its lifetime will be bequeathed to its descendants. Households optimise over their lifetime, based on their expectations on labour earnings they can obtain (which in turns also depends on their ability type) and interest rates on savings.[2] Taxation affects net wages, interest rates and the price of goods, thus it also influences how households and firms behave. The only uncertainty faced by households in the model is due to their mortality risk.

On the production side, one representative perfectly competitive firm maximises static profits generated from the production of a single good by choosing capital and labour demand. Production technology is described by the Cobb-Douglas function. Exogenous productivity growth in the form of labour augmenting technological change is assumed. EDGE-M3 can be optionally run assuming a closed economy or a partially open economy.

The government collects taxes and distributes transfers to the households. In the current version of the model we distinguish labour income tax, payroll tax, capital income taxes (on savings and pension income), and consumption taxes (VAT and excises). The govern-

[1] The model's design was largely inspired by OG-USA, an open-source model for the US economy as described in Evans and DeBacker (2019).

[2] Households are represented by households' heads, i.e. the person with the highest income in a household.

ment grants a general transfer to households that is composed of pension transfers and other transfers. There are two options for the government closure in EDGE-M3: by means of the consumption tax or by means of the other than pension transfers. Since EDGE-M3 is a general equilibrium model, all markets must clear. There are a capital market, a labour market and a goods market in the model. The current version of the model is deterministic, i.e. there are no aggregate shocks.

In terms of calibration, the EDGE-M3 family of overlapping generations models makes use of both macroeconomic and, most importantly, microeconomic data. In this way, the underlying characteristics of the heterogeneous individuals are captured in the macro model. For example, the parameters affecting the disutility of labour supply are calibrated using data on hours worked and the EUROMOD microsimulation model. We match labour elasticities estimated using the micro-data to the more aggregated individual agents of our OLG model to produce a labour supply curve. To obtain realistic consumption profiles, our model's parameters have been calibrated to closely reproduce the actual wealth and bequests distributions in a specific country.

A feature of the model which is of particular interest is the richness of its income tax functions. In order to model taxes and following DeBacker et al. (2019), we equip the EDGE-M3 model with non-linear income tax functions estimated using the output from the EUROMOD microsimulation model. We assume that tax rates on labour income and capital income are bivariate non-linear functions of labour income and capital income. Thanks to this highly disaggregated approach, important characteristics of the complex tax system are automatically accounted for by means of the parameterised tax functions that enter the macro model.[3]
Furthermore, we separately estimate a social insurance contribution function using the underlying microeconomic data. We consider mandatory contributions to pension schemes separately from income taxation due to their nature of forced savings. This approach also enables more controlled policy experiments, for instance in order to examine tax reforms under a *ceteris paribus* condition with respect to the pension system, and vice versa.

In sum, the richness of the EDGE-M3 model makes it highly suitable for a joint analysis of the individual savings and labour responses, the macroeconomic effects as well as the inter- and intragenerational impact of pressing policy questions including, but not limited to:

- demographic change, since EDGE-M3 includes yearly country-specific demographic projections from Eurostat for fertility rates, mortality rates and immigration rates;
- reforms of social insurance contributions or labour and capital income taxes, such as shifts from labour to capital;

[3]The estimation of the income tax functions is detailed in Section 3.4. In d'Andria et al. (2019), we illustrate the methodology of including non-linear income tax functions into the EDGE-M3 model by first examining the effect of a reduction in marginal personal income tax rates in Italy with the EUROMOD microsimulation model and then translating the microsimulation results into a shock for the overlapping generations model.

- distributional concerns, including changes to the progressivity of the tax system, where results may critically hinge on the behaviour of the top share of earners;
- pension system reforms and their distributional impact, e.g. increasing the retirement age, changing the pension contribution rate or switching from one pension system to another one.

The EDGE-M3 is currently being extended to analyse European pension systems, namely the defined benefit, the defined contribution and the point system.

The paper is structured as follows. In Section 2 we discuss the model structure presenting all the equations and assumptions, detailing the functional forms used. Section 3 presents the calibration of the model using both macroeconomic and microeconomic data. Section 4 finally offers some concluding remarks.

2 Model structure

In this section, we present in detail the EDGE-M3 model structure.

2.1 Households

The household is, in many respects, the most important economic agent in the EDGE-M3 model. We model households in EDGE-M3 rather than individuals, because we want to abstract from the concepts of gender, marital status, and number of children.[4] Therefore, it is appropriate to use the household as the most granular unit of account.

2.1.1 Budget Constraint

We described the derivation and dynamics of the population distribution in Section 3.1. A measure $\omega_{1,t}$ of households is born each period, become economically relevant at age $s = E+1$ if they survive to that age, and live for up to $E+S$ periods (S economically active periods), with the population of age-s individuals in period t being $\omega_{s,t}$. Let the age of a household be indexed by $s = \{1, 2, ...E+S\}$.

At birth, each household age $s = 1$ is randomly assigned one of J ability groups, indexed by j. Let λ_j represent the fraction of individuals in each ability group, such that $\sum_j \lambda_j = 1$. Note that this implies that the distribution across ability types in each age is given by $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, ...\lambda_J]$. Once a household member is born and assigned to an ability type, he remains that ability type for his entire lifetime. Thus, it is the deterministic ability heterogeneity as an agent cannot change his ability type (for more details see Section 3.2). Let $e_{j,s} > 0$ be a matrix of ability-levels such that an individual of ability type j will have lifetime abilities of $[e_{j,1}, e_{j,2}, ...e_{j,E+S}]$. The budget constraint for the age-s household in

[4]The curse of dimensionality forces us to focus on those household characteristics – age and ability – most relevant for the analyses considered.

lifetime income group j at time t is the following,

$$c_{j,s,t}\left(1+\tau^c_{s,t}\right) + b_{j,s+1,t+1} = (1+r_t)b_{j,s,t} + w_t e_{j,s}n_{j,s,t} + \zeta_{j,s}\frac{BQ_t}{\lambda_j\omega_{s,t}} + \eta_{j,s,t}\frac{TR_t}{\lambda_j\omega_{s,t}} - T^I_{j,s,t} - T^P_{j,s,t}$$
$$\forall j,t \quad \text{and} \quad s\geq E+1 \quad \text{where} \quad b_{j,E+1,t}=0 \quad \forall j,t \tag{1}$$

where $c_{j,s,t}$ is consumption, $\tau^c_{s,t}$ is consumption tax rate, $b_{j,s+1,t+1}$ is savings for the next period, r_t is the interest rate (return on savings), $b_{j,s,t}$ is current period wealth (savings from last period), w_t is the wage, and $n_{j,s,t}$ is labour supply.

The next term on the right-hand-side of the budget constraint (1) represents the portion of total bequests BQ_t that go to the age-s, income-group-j household. Let $\zeta_{j,s}$ be the fraction of total bequests BQ_t that go to the age-s, income-group-j household, such that $\sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\zeta_{j,s}=1$. We must divide that amount by the population of (j,s) households $\lambda_j\omega_{s,t}$. Section 3.3.2 details how to calibrate the $\zeta_{j,s}$ values from consumer finance data.

The penultimate term on the right-hand-side of the budget constraint (1) represents the portion of total transfers TR_t that go to the age-s, income-group-j household. Let $\eta_{j,s,t}$ be the fraction of total transfers TR_t that go to the age-s, income-group-j household, such that $\sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\eta_{j,s,t}=1$. We must divide that amount by the population of (j,s) households $\lambda_j\omega_{s,t}$. Section 2.3.3 details how transfers are distributed among households.

The last two terms on the right-hand-side of the budget constraint (1) represent income taxes paid by households, $T^I_{j,s,t}$, and payroll tax, $T^P_{j,s,t}$.

2.1.2 Elliptical Disutility of Labour Supply

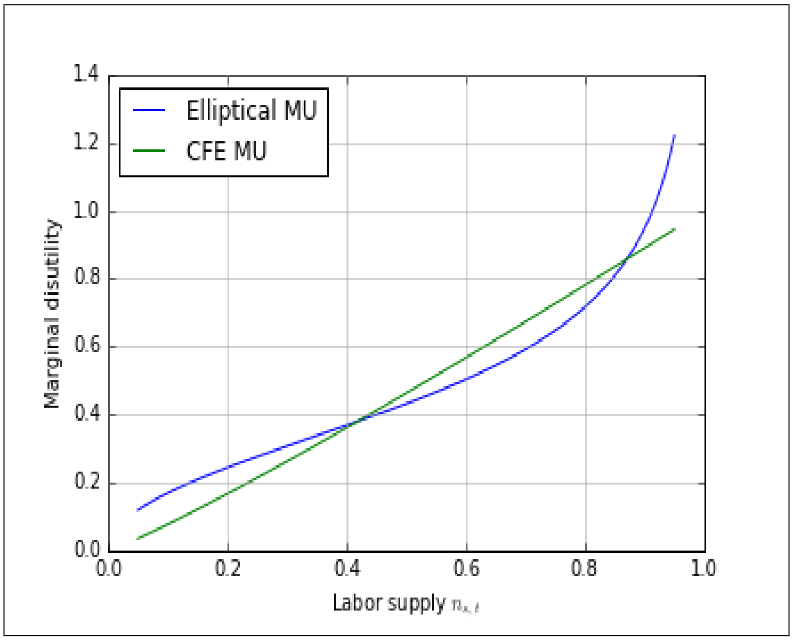
In `EDGE-M3`, the period utility function of each household is a function of consumption $c_{j,s,t}$, savings $b_{j,s+1,t+1}$, and labour supply $n_{j,s,t}$.[5] We detail this utility function, its justification, and functional form in Section 2.1.3. With endogenous labour supply $n_{j,s,t}$, we must specify how labour enters an agent's utility function and what are the constraints. Assume that each household is endowed with a measure of time $\tilde{l}$ each period that it can choose to spend as either labour $n_{j,s,t}\in[0,\tilde{l}]$ or leisure $l_{j,s,t}\in[0,\tilde{l}]$.

$$n_{j,s,t} + l_{j,s,t} = \tilde{l} \quad \forall s,t \tag{2}$$

The functional form for the utility of leisure or the disutility of labour supply has important implications for the computational tractability of the model. One difference of the household's labour supply decision $n_{j,s,t}$ from the consumption decision $c_{j,s,t}$ is that the consumption decision only has a lower bound $c_{j,s,t}\geq 0$ whereas the labour supply decision has both upper and lower bounds $n_{j,s,t}\in[0,\tilde{l}]$. Evans and Phillips (2018) show that many of the traditional functional forms for the disutility of labour—Cobb-Douglas, constant Frisch elasticity, constant relative risk aversion (CRRA)—do not have Inada conditions on both the upper and lower bounds of labour supply. To solve these in a heterogeneous agent model would require occasionally binding constraints, which is a notoriously difficult computational problem.

[5] Savings enters the period utility function to provide a "warm glow" bequest motive.

Figure 1: Comparison of CFE marginal disutility of leisure $\theta = 0.9$ to fitted elliptical utility



Evans and Phillips (2018) propose using an equation for an ellipse to match the disutility of labour supply to whatever traditional functional form one wants. Our preferred specification in `EDGE-M3` is to fit an elliptical disutility of labour supply function to approximate a linearly separable constant Frisch elasticity (CFE) functional form. Let $v(n)$ be a general disutility of labour function. A CFE disutility of labour function is the following,

$$v(n) \equiv \frac{n^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}, \quad \theta > 0 \tag{3}$$

where $\theta > 0$ represents the Frisch elasticity of labour supply. The elliptical disutility of labour supply functional form is the following,

$$v(n) = -b\left[1 - \left(\frac{n}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}}, \quad b, \upsilon > 0 \tag{4}$$

where $b > 0$ is a scale parameter and $\upsilon > 0$ is a curvature parameter. This functional form satisfies both $v'(n) > 0$ and $v''(n) > 0$ for all $n \in (0, 1)$. Further, it has Inada conditions at both the upper and lower bounds of labour supply $\lim_{n\to 0} v'(n) = 0$ and $\lim_{n\to \tilde{l}} v'(n) = -\infty$.

Because it is the marginal disutility of labour supply that matters for household decision making, we want to choose the parameters of the elliptical disutility of labour supply function (b, υ) so that the elliptical marginal utilities match the marginal utilities of the CFE disutility of labour supply. Figure 1 shows the fit of marginal utilities for a Frisch elasticity of $\theta = 0.9$ and a total time endowment of $\tilde{l} = 1.0$. The estimated elliptical utility parameters in this case are $b = 0.527$ and $\upsilon = 1.497$.[6]

[6] Peterman (2016) shows that in a macro-model that has only an intensive margin of labour supply and no

2.1.3 Optimality Conditions

Households choose lifetime consumption $\{c_{j,s,t+s-1}\}_{s=1}^{S}$, labour supply $\{n_{j,s,t+s-1}\}_{s=1}^{S}$, and savings $\{b_{j,s+1,t+s}\}_{s=1}^{S}$ to maximise lifetime utility, subject to the budget constraints and non negativity constraints. The household period utility function is the following.

$$
\begin{split}
u(c_{j,s,t},n_{j,s,t},b_{j,s+1,t+1}) \equiv \frac{(c_{j,s,t})^{1-\sigma}-1}{1-\sigma} + e^{g_y t(1-\sigma)}\chi^n_s\left(b\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}}\right) + \\
\chi^b_j\rho_s\frac{(b_{j,s+1,t+1})^{1-\sigma}-1}{1-\sigma} \quad \forall j,t \quad \text{and} \quad E+1\leq s\leq E+S
\end{split}
\tag{5}
$$

The period utility function (5) is linearly separable in $c_{j,s,t}$, $n_{j,s,t}$, and $b_{j,s+1,t+1}$. The first term is a constant relative risk aversion (CRRA) utility of consumption. The second term is the elliptical disutility of labour described in Section 2.1.2. The constant χ^n_s adjusts the disutility of labour supply relative to consumption and can vary by age s, which is helpful for calibrating the model to match labour market moments. See Section 3.4 for a discussion of the calibration.

It is necessary to multiply the disutility of labour in (5) by $e^{g_y(1-\sigma)}$ because labour supply $n_{j,s,t}$ is stationary, but both consumption $c_{j,s,t}$ and savings $b_{j,s+1,t+1}$ are growing at the rate of technological progress (see Section 2.5). The $e^{g_y(1-\sigma)}$ term keeps the relative utility values of consumption, labour supply, and savings in the same units.

The final term in the period utility function (5) is the "warm glow" bequest motive. It is a CRRA utility of savings, discounted by the mortality rate ρ_s.[7] Intuitively, it represents the utility a household gets in the event that they don't live to the next period with probability ρ_s. It is a utility of savings beyond its usual benefit of allowing for more consumption in the next period. This utility of bequests also has constant χ^b_j which adjusts the utility of bequests relative to consumption and can vary by lifetime income group j. This is helpful for calibrating the model to match wealth distribution moments. See Section 3.4 for a discussion of the calibration. Note that any bequest before age $E+S$ is unintentional as it was bequeathed due an event of death that was uncertain. Intentional bequests are all bequests given in the final period of life in which death is certain $b_{j,E+S+1,t}$.

The household lifetime optimisation problem is to choose consumption $c_{j,s,t}$, labour supply $n_{j,s,t}$, and savings $b_{j,s+1,t+1}$ in every period of life to maximise expected discounted lifetime utility, subject to budget constraints and upper-bound and lower-bound constraints.

$$
\max_{\{(c_{j,s,t}),(n_{j,s,t}),(b_{j,s+1,t+1})\}_{s=E+1}^{E+S}} \sum_{s=1}^{S}\beta^{s-1}\left[\Pi_{u=E+1}^{E+s}(1-\rho_u)\right]u(c_{j,s,t+s-1},n_{j,s,t+s-1},b_{j,s+1,t+s}) \tag{6}
$$

$$
\text{s.t.} \quad c_{j,s,t}\left(1+\tau^c_{s,t}\right)+b_{j,s+1,t+1} = (1+r_t)b_{j,s,t}+w_t e_{j,s}n_{j,s,t}+\zeta_{j,s}\frac{BQ_t}{\lambda_j\omega_{s,t}}+\eta_{j,s,t}\frac{TR_t}{\lambda_j\omega_{s,t}}-T^I_{j,s,t}-T^P_{j,s,t} \tag{1}
$$

$$
\text{and} \quad c_{j,s,t}\geq 0,\; n_{j,s,t}\in[0,\tilde{l}],\ \text{and}\ b_{j,E+1,t}=0 \quad \forall j,t,\ \text{and}\ E+1\leq s\leq E+S
$$

---

extensive margin and represents a broad composition of individuals supplying labour—such as EDGE-M3—a Frisch elasticity of around 0.9 is probably appropriate. He tests the implied macro elasticity when the assumed micro elasticities are small on the intensive margin but only macro aggregates—which include both extensive and intensive margin agents—are observed.

[7] See Section 3.1.2 of Section 3.1 for a detailed discussion of mortality rates in EDGE-M3.

The non-negativity constraint on consumption does not bind in equilibrium because of the Inada condition $\lim_{c\to 0} u_1(c, n, b') = \infty$, which implies consumption is always strictly positive in equilibrium $c_{j,s,t} > 0$ for all j, s, and t. The warm glow bequest motive in (5) also has an Inada condition for savings at zero, so $b_{j,s,t} > 0$ for all j, s, and t. This is an implicit borrowing constraint.[8] And finally, as discussed in Section 2.1.2, the elliptical disutility of labour supply functional form in (5) imposes Inada conditions on both the upper and lower bounds of labour supply such that labour supply is strictly interior in equilibrium $n_{j,s,t} \in (0, \tilde{l})$ for all j, s, and t.

The household maximisation problem can be further reduced by substituting in the household budget constraint, which binds with equality. This simplifies the household's problem to choosing labour supply $n_{j,s,t}$ and savings $b_{j,s+1,t+1}$ every period to maximise lifetime discounted expected utility. The $2S$ first order conditions for every type-j household that characterise the its S optimal labour supply decisions and S optimal savings decisions are the following.

$$\left(w_t e_{j,s} - \frac{\partial T^I_{j,s,t}}{\partial n_{j,s,t}} - \frac{\partial T^P_{j,s,t}}{\partial n_{j,s,t}}\right)(c_{j,s,t})^{-\sigma}\left(\frac{1}{1+\tau^c_{s,t}}\right) = e^{g_y(1-\sigma)}\chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}$$
$$\forall j,t, \quad \text{and} \quad E+1 \leq s \leq E+S \tag{7}$$

$$(c_{j,s,t})^{-\sigma}\left(\frac{1}{1+\tau^c_{s,t}}\right) = e^{-g_y\sigma}\left(\chi^b_j\rho_s(b_{j,s+1,t+1})^{-\sigma} + \beta\big(1-\rho_s\big)\left[1 + r_{t+1} - \frac{\partial T^I_{j,s+1,t+1}}{\partial b_{j,s+1,t+1}}\right](c_{j,s+1,t+1})^{-\sigma}\left(\frac{1}{1+\tau^c_{s+1,t+1}}\right)\right)$$
$$\forall j,t, \quad \text{and} \quad E+1 \leq s \leq E+S-1 \tag{8}$$

$$(c_{j,E+S,t})^{-\sigma} = \chi^b_j(b_{j,E+S+1,t+1})^{-\sigma} \quad \forall j,t \quad \text{and} \quad s = E+S \tag{9}$$

where the marginal income tax rate with respect to labour supply $\frac{\partial T_{s,t}}{\partial n_{j,s,t}}$ is described in equation (28). $\frac{\partial T^P_{j,s,t}}{\partial n_{j,s,t}}$ is the marginal rate of payroll tax with respect to labour supply.

2.1.4 Expectations

To conclude the household's problem, we must make an assumption about how the age-s household can forecast the time path of interest rates, wages, and total bequests $\{r_u, w_u, BQ_u\}_{u=t}^{t+S-s}$ over his remaining lifetime. As shown in Appendices B.1 and B.2, the equilibrium interest

[8] It is important to note that savings also has an implicit upper bound $b_{j,s,t} \leq k$ above which consumption would be negative in current period. However, this upper bound on savings is taken care of by the Inada condition on consumption.

rate r_t, wage w_t, and total bequests BQ_t will be functions of the state vector $\mathbf{\Gamma}_t$, which turns out to be the entire distribution of savings in period t.

Define $\mathbf{\Gamma}_t$ as the distribution of household savings across households at time t.

$$\mathbf{\Gamma}_t \equiv \left\{b_{j,s,t}\right\}_{s=E+2}^{E+S} \quad \forall j, t \tag{10}$$

Let general beliefs about the future distribution of capital in period $t+u$ be characterised by the operator $\Omega(\cdot)$ such that:

$$\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1 \tag{11}$$

where the e superscript signifies that $\mathbf{\Gamma}^e_{t+u}$ is the expected distribution of wealth at time $t+u$ based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.[9]

2.2 Firms

The production side of the `EDGE-M3` model is populated by a unit measure of identical perfectly competitive firms that rent capital K_t and hire labour L_t to produce output Y_t.

2.2.1 Production Function

Firms produce output Y_t using inputs of capital K_t and labour L_t according to the Cobb-Douglas production function:

$$Y_t = Z_t(K_t)^{\gamma}(e^{g_y t}L_t)^{1-\gamma} \tag{12}$$

where Z_t is an exogenous scale parameter (total factor productivity) that can be time dependent and γ represents the capital share of income. We have included constant productivity growth g_y as the rate of labour augmenting technological progress.
The Cobb-Douglas production function is a special case of the general constant elasticity (CES) of substitution production function,

$$Y_t = F(K_t, L_t) \equiv Z_t\left[(\gamma)^{\frac{1}{\varepsilon}}(K_t)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)^{\frac{1}{\varepsilon}}(e^{g_y t}L_t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \quad \forall t \tag{13}$$

for $\varepsilon = 1$.

2.2.2 Optimality Conditions

The profit function of the representative firm is the following.

$$PR_t = F(K_t, L_t) - w_t L_t - \big(r_t + \delta\big)K_t \quad \forall t \tag{14}$$

Gross income for the firms is given by the production function $F(K, L)$ because we have normalised the price of the consumption good to 1. Labour costs to the firm are $w_t L_t$, and capital costs are $(r_t + \delta)K_t$. The per-period economic depreciation rate is given by δ.

[9] In Appendix B.2 we will assume that beliefs are correct (rational expectations) for the non-steady-state equilibrium in Definition 2.

Taking the derivative of the profit function (14) with respect to labour L_t and setting it equal to zero and taking the derivative of the profit function with respect to capital K_t and setting it equal to zero, respectively, characterises the optimal labour and capital demands.

$$w_t = (1-\gamma)\frac{Y_t}{L_t} \quad \forall t \tag{15}$$

$$r_t = \gamma\frac{Y_t}{K_t} - \delta \quad \forall t \tag{16}$$

2.2.3 Small Open Economy

In addition to a closed economy version, the `EDGE-M3` model also accommodates small and partially open economies. In the small open economy version of the `EDGE-M3` model, the country faces an exogenous world interest rate, r_t^* that determines the amount of savings and investment. If the supply of savings from households does not meet the demand for private capital and private borrowing, foreign capital will flow in to make excess demand zero at the world interest rate.

Let the total capital stock be given by the quantity of domestically supplied capital and foreign supplied capital, i.e., $K_t = K_t^d + K_t^f$. Then foreign capital is given by:

$$K_t^f = K_t^{demand} - (B_t - D_t) \tag{17}$$

where B_t is aggregate household savings and D_t is government borrowing. Capital demand is determined from the firm's first order condition (16) for its choice of capital, given r_t^*.

2.2.4 Partially Open Economy

In the partially open economy version of `EDGE-M3`, the openness of the economy is modelled through two parameters that capture the extent of foreign lending to the domestic government and the amount of foreign lending of private capital to firms.

First, this version of the model accommodates foreign held public debt. In particular, the parameter ζ_D gives the share of new debt issues that are purchased by foreigners. The law of motion for foreign-held debt is therefore given by:

$$D_{t+1}^f = D_t^f + \zeta_D(D_{t+1} - D_t) \tag{18}$$

Domestic debt holdings then are determined as the remaining debt holdings needed to meet government demand for debt:

$$D_t^d = D_t - D_t^f \tag{19}$$

Second, whereas total capital demand still follows from the exogenous world interest rate, the parameters ζ_K helps to determine the share of domestic capital held by foreigners. In particular, let K_t^{open} be the amount of capital that would need to flow into the country to meet firm demand for capital at the exogenous world interest rate from the small open economy specification, net of what domestic households can supply:

$$K_t^{open} = K_t^{demand} - (B_t - D_t) \tag{20}$$

where, K_t^{demand} is total capital demand by domestic firms at r_t^*, B_t are total asset holdings of domestic households, and D_t are holdings of government debt by domestic households. Importantly, total asset holdings from households result from solving the household's optimisation problem at the endogenous home country interest rate. Note that there thus is a disconnect between the interest rates that determine firm capital demand and domestic household savings and the interest rate used to determine K_t^{demand}. This assumption is useful in that it nests the small open economy case into the partial open economy model. However, it does leave out the realistic responses of foreign capital supply to differentials in the home country interest rate and the world interest rate.

Next, given K_t^{open}, ζ_K can be used to determine the foreign capital held by foreigners in the small open economy specification:

$$K_t^f = \zeta_K K_t^{open} \tag{21}$$

Given the two equations above, we can find the total supply of capital as:

$$\begin{aligned} K_t^{supply} &= K_t^d + K_t^f \\ &= B_t - D_t^d + \zeta_K K_t^{open} \end{aligned} \tag{22}$$

2.3 Government — Household Taxes and Transfers

The government is not an optimising agent in `EDGE-M3`. The government levies taxes on households and provides transfers to households. The government sector influences households through three terms in the household budget constraint given by formula (1): government transfers TR_t, total income tax liability function $T_{s,t}^I$, which can be decomposed into the effective income tax rate[10] times total income (see equation (25)), and the consumption tax rate. In this section, we detail the household tax component of government activity $T_{s,t}^I$ in `EDGE-M3`, along with our method of incorporating detailed microsimulation data into a dynamic general equilibrium model. Finally, this section discusses the government's resulting budget constraint.

2.3.1 Income Taxation

Incorporating realistic tax and incentive detail into a general equilibrium model is notoriously difficult for two reasons. First, it is impossible in a dynamic general equilibrium (DGE) model to capture all of the dimensions of heterogeneity on which the real-world tax rate depends. For example, a household's tax liability in reality depends on filing status, number of dependants, many types of income, and some characteristics correlated with age. A good heterogeneous agent DGE model tries to capture the most important dimensions of heterogeneity, and necessarily neglects the other dimensions.

The second difficulty in modelling realistic tax and incentive detail is the need for good microeconomic data on the individuals who make up the economy from which to simulate behavioural responses and corresponding tax liabilities and tax rates.

[10] In this paper effective tax rate refers to the average effective tax rate.

EDGE-M3 follows the method of DeBacker et al. (2019) of generating detailed income tax data on effective tax rates and marginal tax rates for a sample of tax filers along with their respective income and demographic characteristics and then using that data to estimate parametric tax functions that can be incorporated into EDGE-M3 .

Effective and Marginal Tax Rates Before going into more detail regarding how we handle these two difficulties in EDGE-M3 , we need to define some functions and make some notation. For notational simplicity, we will use the variable x to summarise labour income, and we will use the variable y to summarise capital income.

$$x_{j,s,t} \equiv w_t e_{j,s} n_{j,s,t} \quad \forall j,t \quad \text{and} \quad E+1 \leq s \leq E+S \tag{23}$$

$$y_{j,s,t} \equiv r_t b_{j,s,t} \quad \forall j,t \quad \text{and} \quad E+1 \leq s \leq E+S \tag{24}$$

Part of total tax liability $T_{j,s,t}$ from the household budget constraint 1 is income tax liability $T^I_{j,s,t}$ that can be expressed as an effective tax rate multiplied by total income.

$$T^I_{j,s,t} = \tau^{etr}_{s,t}\left(x_{j,s,t}, y_{j,s,t}\right)\left(x_{j,s,t} + y_{j,s,t}\right) \tag{25}$$

Rearranging equation (25) gives the definition of an effective tax rate (ETR) as total income tax liability divided by the unadjusted gross income, or rather, total income tax liability as a percent of unadjusted gross income. A marginal income tax rate (MTR) is defined as the change in total tax liability from a small change income. In EDGE-M3 , we differentiate between the marginal tax rate on labour income ($MTRx$) and the marginal tax rate on labour income ($MTRy$).

$$\tau^{mtrx} \equiv \frac{\partial T^I_{j,s,t}}{\partial w_t e_{j,s} n_{j,s,t}} = \frac{\partial T^I_{j,s,t}}{\partial x_{j,s,t}} \quad \forall j,t \quad \text{and} \quad E+1 \leq s \leq E+S \tag{26}$$

$$\tau^{mtry} \equiv \frac{\partial T^I_{j,s,t}}{\partial r_t b_{j,s,t}} = \frac{\partial T^I_{j,s,t}}{\partial y_{j,s,t}} \quad \forall j,t \quad \text{and} \quad E+1 \leq s \leq E+S \tag{27}$$

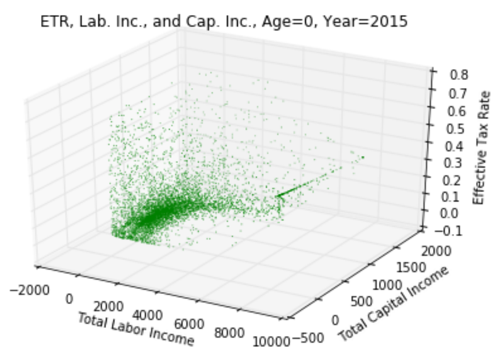
As we show in Section 2.1.3, the derivative of total income tax liability with respect to labour supply $\frac{\partial T^I_{s,t}}{\partial n_{j,s,t}}$ and the derivative of total income tax liability next period with respect to savings $\frac{\partial T^I_{s+1,t+1}}{\partial b_{j,s+1,t+1}}$ show up in the household Euler equations for labour supply given in equation (7) and savings given in equation (8), respectively. It is valuable to be able to express those marginal tax rates, for which we have no data, as marginal tax rates for which we do have data. The following two expressions show how the marginal tax rates of labour supply can be expressed as the marginal tax rate on labour income times the household-specific wage and how the marginal tax rate of savings can be expressed as the marginal tax rate of capital income times the interest rate.

$$\frac{\partial T^I_{j,s,t}}{\partial n_{j,s,t}} = \frac{\partial T^I_{j,s,t}}{\partial w_t e_{j,s} n_{j,s,t}} \frac{\partial w_t e_{j,s} n_{j,s,t}}{\partial n_{j,s,t}} = \frac{\partial T^I_{j,s,t}}{\partial w_t e_{j,s} n_{j,s,t}} w_t e_{j,s} = \tau^{mtrx}_{s,t} w_t e_{j,s} \tag{28}$$

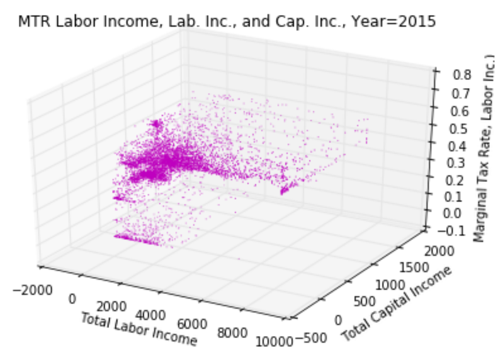
$$\frac{\partial T^I_{j,s,t}}{\partial b_{j,s,t}} = \frac{\partial T^I_{j,s,t}}{\partial r_t b_{j,s,t}} \frac{\partial r_t b_{j,s,t}}{\partial b_{j,s,t}} = \frac{\partial T^I_{j,s,t}}{\partial r_t b_{j,s,t}} r_t = \tau^{mtry}_{s,t} r_t \tag{29}$$

Fitting Tax Functions In looking at the 3D scatter plots of ETR, $MTRx$, and $MTRy$ in Figure 2, it is clear that all of these rates exhibit negative exponential or logistic shape. This empirical regularity allows us to make an important and non-restrictive assumption. We can fit parametric income tax rate functions to these data that are constrained to be monotonically increasing in labour income and capital income. This assumption of monotonicity is computationally important as it preserves a convex budget set for each household, which is important for being able to solve many household lifetime problems over a large number of periods.

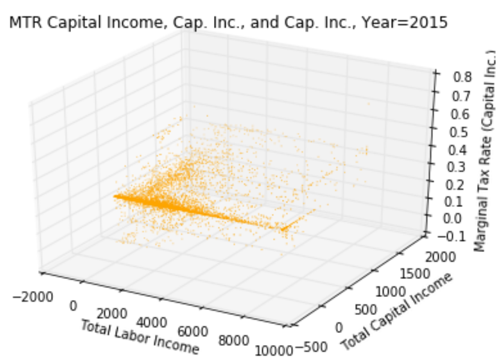
Figure 2: Scatter plot of ETR, MTRx, MTRy, and histogram as functions of labour income and capital income from microsimulation model, year 2015



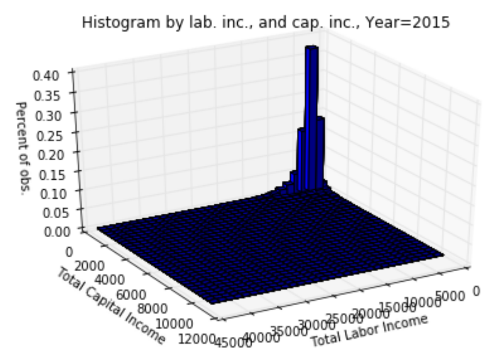
(a) Effective tax rates ETR



(b) Marginal tax rates on labour income $MTRx$



(c) Marginal tax rates on capital income $MTRy$



(d) Histogram

EDGE-M3 follows the approach of DeBacker et al. (2019) in using the functional form expressed by equation (30) to estimate income tax functions for each time period t. The estimation can be performed separately for each age $s = E + 1, E + 2, ..., E + S$, or dividing observations into age bins and then estimating functions for each bin. The option to have age-dependent estimations comes useful when the data displays heterogeneous compositions in incomes that vary with age. As different income items are often treated differently tax-wise, for example in the case of property or pension income, having age-specific tax functions allows to indirectly capture such heterogeneity thus providing better estimates. Age-dependent tax functions and the size in years of each age bin for which the estimation is performed can be changed via parameters, in order to be able to deal with the specific micro data used for this purpose.

Equation (30) is written as a generic tax rate, but we use this same functional form for $ETR's$, $MTRx's$, and $MTRy's$.

$$
\begin{aligned}
\tau(x,y) = &\left[\tau(x) + shift_x\right]^{\phi}\left[\tau(y) + shift_y\right]^{1-\phi} + shift \\
&\text{where} \quad \tau(x) \equiv (max_x - min_x)\left(\frac{Ax^2 + Bx}{Ax^2 + Bx + 1}\right) + min_x \\
&\text{and} \quad \tau(y) \equiv (max_y - min_y)\left(\frac{Cy^2 + Dy}{Cy^2 + Dy + 1}\right) + min_y \\
&\text{where} \quad A, B, C, D, max_x, max_y, shift_x, shift_y > 0 \quad \text{and} \quad \phi \in [0,1] \\
&\text{and} \quad max_x > min_x \quad \text{and} \quad max_y > min_y
\end{aligned}
\tag{30}
$$

The parameters values will, in general, differ across the different functions (effective and marginal rate functions) and by age, s, and tax year, t. We drop the subscripts for age and year from the above exposition for clarity.

By assuming each tax function takes the same form, we are breaking the analytical link between the the effective tax rate function and the marginal rate functions. In particular, one could assume an effective tax rate function and then use the analytical derivative of that to find the marginal tax rate function. However, we have found it useful to separately estimate the marginal and average rate functions. One reason is that we want the tax functions to be able to capture policy changes that have differential effects on marginal and average rates. For example, a change in the standard deduction for tax payers would have a direct effect on their average tax rates. But it will have secondary effect on marginal rates as well, as some filers will find themselves in different tax brackets after the policy change. These are smaller and second order effects. When tax functions are fit to the new policy, in this case a lower standard deduction, we want them to be able to represent this differential impact on the marginal and average tax rates. The second reason is related to the first. As the additional flexibility allows us to model specific aspects of tax policy more closely, it also allows us to better fit the parameterised tax functions to the data.

The key building blocks of the functional form equation (30) are the $\tau(x)$ and $\tau(y)$ univariate functions. The ratio of polynomials in the $\tau(x)$ function with positive $\frac{Ax^2+Bx}{Ax^2+Bx+1}$ coefficients A, $B > 0$ and positive support for labour income $x > 0$ creates a negative-exponential-shaped function that is bounded between 0 and 1, and the curvature is governed by the ratio of quadratic polynomials. The multiplicative scalar term $(max_x - min_x)$ on the ratio of polynomials and the addition of min_x at the end of $\tau(x)$ expands the range of the univariate negative-exponential-shaped function to $\tau(x) \in [min_x, max_x]$. The $\tau(y)$ function is an analogous univariate negative-exponential-shaped function in capital income y, such that $\tau(y) \in [min_y, max_y]$.

The respective $shift_x$ and $shift_y$ parameters in equation (30) are analogous to the additive constants in a Stone-Geary utility function. These constants ensure that the two sums $\tau(x) + shift_x$ and $\tau(yx) + shift_y$ are both strictly positive. They allow for negative tax rates in the $\tau(.)$ functions despite the requirement that the arguments inside the brackets be strictly positive. The general shift parameter outside of the Cobb-Douglas brackets can then shift the tax rate function so that it can accommodate negative tax rates. The Cobb-Douglas share parameter $\phi \in [0, 1]$ controls the shape of the function between the two univariate functions $\tau(x)$ and $\tau(y)$.

This functional form for tax rates delivers flexible parametric functions that can fit the tax rate data shown in Figure 2 as well as a wide variety of policy reforms. Further, these functional forms are monotonically increasing in both labour income x and capital income y. This characteristic of monotonicity in x and y is essential for guaranteeing convex budget sets and thus uniqueness of solutions to the household Euler equations. The assumption of monotonicity does not appear to be a strong one when viewing the tax rate data shown in Figure 2. While it does limit the potential tax systems to which one could apply our methodology, tax policies that do not satisfy this assumption would result in non-convex budget sets and thus require non-standard DGE model solutions methods and would not guarantee a unique equilibrium. The 12 parameters of our tax rate functional form from equation (30) are summarised in Table 1.

Table 1: Description of tax rate function $\tau(x, y)$ parameters

Symbol	Description
A	Coefficient on squared labour income term x^2 in $\tau(x)$
B	Coefficient on labour income term x in $\tau(x)$
C	Coefficient on squared capital income term y^2 in $\tau(y)$
D	Coefficient on capital income term y in $\tau(y)$
max_x	Maximum tax rate on labour income x given $y = 0$
min_x	Minimum tax rate on labour income x given $y = 0$
max_y	Maximum tax rate on capital income y given $x = 0$
min_y	Minimum tax rate on capital income y given $x = 0$
$shift_x$	$shifter > \lvert min_x \rvert$ ensures that $\tau(x) + shift_x > 0$ despite potentially negative values for $\tau(x)$
$shift_y$	$shifter > \lvert min_y \rvert$ ensures that $\tau(y) + shift_y > 0$ despite potentially negative values for $\tau(y)$
$shift$	shifter (can be negative) allows for support of $\tau(x, y)$ to include negative tax rates
ϕ	Cobb-Douglas share parameter between 0 and 1

Source: DeBacker et al. (2019)

Factor Transforming Income Units The income tax functions τ^{etr}, τ^{mtrx}, τ^{mtry} are estimated based on current Italian tax filer reported incomes in Euros. However, the consumption units of the `EDGE-M3` model are not in the same units as the real-world Italian income data. For this reason, we have to transform the income by a $factor$ so that it is in the same units as the income data on which the tax functions were estimated.

The tax rate functions are each functions of capital income and labour income $\tau(x,y)$. In order to make the tax functions return accurate tax rates associated with the correct levels of income, we multiply the model income x^m and y^m by a $factor$. The $factor$ translates model units into the data units (Euros). Thus, we need to multiply model taxable income by a factor to get the same units as the real-world Italian income in tax data $\tau(factor \times x^m, factor \times y^m)$. We define the $factor$ such that average steady-state household total income in the model times the $factor$ equals the Italian tax data average total income.

$$factor\left[\sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\lambda_j\omega_s\left(we_{j,s}n_{j,s}+rb_{j,s}\right)\right] = \quad \text{average household income in tax data} \tag{31}$$

We do not know the steady-state wage, interest rate, household labour supply, and savings *ex ante*. So the income $factor$ is an endogenous variable in the steady-state equilibrium computational solution. We hold the factor constant throughout the non-steady-state equilibrium solution.

2.3.2 Consumption Tax

Consumption tax is the average tax paid on all goods and services, including both value-added tax and excise duties. in `EDGE-M3` consumption tax rates are differentiated by age. In Section 3.4.3 we discuss how the consumption tax rates are calibrated in `EDGE-M3` using micro data.

2.3.3 Household Transfers

Total transfers to households by the government in a given period t is TR_t. The percent of those transfers given to all households of age s and lifetime income group j is $\eta_{j,s}$ such that $\sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\eta_{j,s,t}=1$. In the current calibration `EDGE-M3` has the transfer distribution function set to distribute transfers uniformly among the population.

$$\eta_{j,s,t}=\frac{\lambda_j\omega_{s,t}}{\tilde{N}_t}\quad\forall j,t\quad\text{and}\quad E+1\leq s\leq E+S \tag{32}$$

However, this distribution function $\eta_{j,s,t}$ could also be modified to more accurately reflect the way transfers are distributed in Italy.

2.3.4 Government Budget Constraint

Let the level of government debt in period t be given by D_t. The government budget constraint requires that government revenues Rev_t from income and consumption taxes plus the budget deficit $(D_{t+1}-D_t)$ equal expenditures on interest of the debt and government spending on transfer payments to households TR_t every period t.

$$D_{t+1} + Rev_t = (1 + r_{gov,t})D_t + TR_t \quad \forall t \tag{33}$$

Despite the model having no aggregate risk, it may be helpful to build in an interest rate differential between the rate of return on private capital and the interest rate on government debt. Doing so helps to add realism by including a risk premium. `EDGE-M3` allows users to set an exogenous wedge between these two rates:

$$r_{gov,t} = (1 - \tau_{d,t})r_t - \mu_d \tag{34}$$

The two parameters, $\tau_{d,t}$ and μ_d can be used to allow for a government interest rate ($r_{gov,t}$) that is a percentage hair cut from the market rate or a government interest rate with a constant risk premium.

In the cases where there is a differential ($\tau_{d,t}$ or $\mu_d \neq 0$), then we need to be careful to specify how the household chooses government debt and private capital in its portfolio of asset holdings. We make the assumption that under the exogenous interest rate wedge, the household is indifferent between holding its assets as debt and private capital. This amounts to an assumption that these two assets are perfect substitutes given the exogenous wedge in interest rates. Given the indifference between government debt and private capital at these two interest rates, we assume that the household holds debt and capital in the same ratio that debt and capital are demanded by the government and private firms, respectively. The interest rate on the household portfolio of asset is thus given by:

$$r_{hh,t} = \frac{r_{gov,t}D_t + r_tK_t}{D_t + K_t} \tag{35}$$

2.4 Market Clearing

Three markets must clear in `EDGE-M3`—the labour market, the capital market, and the goods market. By Walras' Law, we only need to use two of those market clearing conditions because the third one is redundant. In the model, we choose to use the labour market clearing condition and the capital market clearing condition. The (redundant) goods market clearing condition—sometimes referred to as the resource constraint—is used as a check on the solution method. We present all three market clearing conditions here.

We also characterise here the law of motion for total bequests BQ_t. Although it is not technically a market clearing condition, one could think of the bequests' law of motion as the bequests' market clearing condition.

2.4.1 Market Clearing Conditions

Labour market clearing (see equation (36) below) requires that aggregate labour demand L_t measured in efficiency units equal the sum of household efficiency labour supplied $e_{j,s}n_{j,s,t}$.

$$L_t = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\omega_{s,t}\lambda_j e_{j,s}n_{j,s,t} \quad \forall t \tag{36}$$

Capital market clearing (see (37)) requires that aggregate capital demand from firms K_t equal the sum of capital savings and investment by households $b_{j,s,t}$.

$$K_t = \sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\Big(\omega_{s-1,t-1}\lambda_j b_{j,s,t} + i_s\omega_{s,t-1}\lambda_j b_{j,s,t}\Big) \quad \forall t \tag{37}$$

Aggregate consumption C_t is defined as the sum of all household consumptions, and aggregate investment is defined by the resource constraint $Y_t = C_t + I_t$ as shown in equation (38).

$$\begin{split} &Y_t = C_t + K_{t+1} - \left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J} i_s\omega_{s,t}\lambda_j b_{j,s,t+1}\right) - (1-\delta)K_t \quad \forall t \\ &\text{where} \quad C_t \equiv \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\omega_{s,t}\lambda_j c_{j,s,t} \end{split} \tag{38}$$

Note that the extra terms with the immigration rate i_s in the capital market clearing equation (37) and the resource constraint (38) accounts for the assumption that age-s immigrants in period t bring with them (or take with them in the case of out-migration) the same amount of capital as their domestic counterparts of the same age. Note also that the term in parentheses with immigration rates i_s in the sum acts is equivalent to a net exports term in the standard equation $Y = C + I + G + NX$. That is, if immigration rates are positive, then immigrants are bringing capital into the country and the term in parentheses has a negative sign in front of it. Negative exports are imports.

2.4.2 Total Bequests Law of Motion

Total bequests BQ_t are the collection of savings of household from the previous period who died at the end of the period. These savings are augmented by the interest rate because they are returned after being invested in the production process.

$$BQ_t = (1+r_t)\left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\rho_{s-1}\lambda_j\omega_{s-1,t-1}b_{j,s,t}\right) \quad \forall t \tag{39}$$

Because the form of the period utility function in (5) ensures that $b_{j,s,t} > 0$ for all j, s, and t, total bequests will always be positive $BQ_{j,t} > 0$ for all j and t.

2.4.3 Total Transfers Law of Motion

Total transfers to households TR_t are the collection of all taxes paid by households, i.e. income taxes, $T^I_{j,s,t}$ and consumption taxes, $T^C_{j,s,t}$.

$$TR_t = \left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\lambda_j\omega_{s-1,t-1}T^I_{j,s,t} + \sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\lambda_j\omega_{s-1,t-1}T^C_{j,s,t}\right) \quad \forall t. \tag{40}$$

2.5 Stationarisation

The previous sections derive all the equations necessary to solve for the steady-state and non-steady-state equilibria of this model. However, because labour productivity is growing at rate g_y as can be seen in the firms' production function (13) and the population is growing at rate $\tilde{g}_{n,t}$ as defined in (58), the model is not stationary. Different endogenous variables of the model are growing at different rates.

Table 2 lists the definitions of stationary versions of these endogenous variables. Variables with a " ˆ " signify stationary variables. The first column of variables are growing at the productivity growth rate g_y. These variables are most closely associated with individual variables. The second column of variables are growing at the population growth rate $\tilde{g}_{n,t}$. These variables are most closely associated with population values. The third column of variables are growing at both the productivity growth rate g_y and the population growth rate $\tilde{g}_{n,t}$. These variables are most closely associated with aggregate variables. The last column shows that the interest rate r_t and household labour supply $n_{j,s,t}$ are already stationary.

Table 2: Stationary variable definitions

Sources of growth			Not
$e^{g_y t}$	$\tilde{N}_t$	$e^{g_y t}\tilde{N}_t$	growing[a]
$\hat{c}_{j,s,t} \equiv \frac{c_{j,s,t}}{e^{g_y t}}$	$\hat{\omega}_{s,t} \equiv \frac{\omega_{s,t}}{\tilde{N}_t}$	$\hat{Y}_t \equiv \frac{Y_t}{e^{g_y t}\tilde{N}_t}$	$n_{j,s,t}$
$\hat{b}_{j,s,t} \equiv \frac{b_{j,s,t}}{e^{g_y t}}$	$\hat{L}_t \equiv \frac{L_t}{\tilde{N}_t}$	$\hat{K}_t \equiv \frac{K_t}{e^{g_y t}\tilde{N}_t}$	r_t
$\hat{w}_t \equiv \frac{w_t}{e^{g_y t}}$		$\hat{BQ}_{j,t} \equiv \frac{BQ_{j,t}}{e^{g_y t}\tilde{N}_t}$	
$\hat{y}_{j,s,t} \equiv \frac{y_{j,s,t}}{e^{g_y t}}$		$\hat{TR}_t \equiv \frac{TR_t}{e^{g_y t}\tilde{N}_t}$	
$\hat{T}^I_{j,s,t} \equiv \frac{T^I_{j,s,t}}{e^{g_y t}}$		$\hat{C}_t \equiv \frac{C_t}{e^{g_y t}\tilde{N}_t}$	
$\hat{T}^C_{j,s,t} \equiv \frac{T^C_{j,s,t}}{e^{g_y t}}$			

[a] The interest rate r_t in ((16)) is already stationary because Y_t and K_t grow at the same rate. Household labour supply $n_{j,s,t} \in [0, \tilde{l}]$ is stationary.

The usual definition of equilibrium would be allocations and prices such that households optimise (7), (8), and (9), firms optimise (15) and (16), and markets clear (36) and (37), and (39). In this section, we show how to stationarise each of these characterising equations so that we can use our fixed point methods described in Sections B.1.1 and B.2.1 to solve for the equilibria in Definitions 1 and 2.

2.5.1 Stationarised Household Equations

The stationary version of the household budget constraint (1) is found by dividing both sides of the equation by $e^{g_y t}$. For the savings term $b_{j,s+1,t+1}$, we must multiply and divide by

$e^{g_y(t+1)}$, which leaves an $e^{g_y} = \frac{e^{g_y(t+1)}}{e^{g_y t}}$ in front of the stationarised variable.

$$\hat{c}_{j,s,t}\left(1+\tau^c_{s,t}\right) + e^{g_y}\hat{b}_{j,s+1,t+1} = (1+r_t)\hat{b}_{j,s,t} + \hat{w}_t e_{j,s} n_{j,s,t} + \zeta_{j,s}\frac{\hat{BQ}_t}{\lambda_j\hat{\omega}_{s,t}} + \eta_{j,s}\frac{\hat{TR}_t}{\lambda_j\hat{\omega}_{s,t}} - \hat{T}^I_{j,s,t} - \hat{T}^P_{j,s,t}$$
$$\forall j,t \quad \text{and} \quad s\geq E+1 \quad \text{where} \quad b_{j,E+1,t}=0 \quad \forall j,t \tag{41}$$

Because total bequests BQ_t grows at both the labour productivity growth rate and the population growth rate, we have to multiply and divide that term by the economically relevant population $\tilde{N}_t$. This stationarises total bequests $\hat{BQ}_t$ and the population level in the denominator $\hat{\omega}_{s,t}$.

We stationarise the Euler equations for labour supply (7) by dividing both sides by $e^{g_y(1-\sigma)}$. On the left-hand-side, e^{g_y} stationarises the wage $\hat{w}_t$ and $e^{-\sigma g_y}$ goes inside the parentheses and stationarises consumption $\hat{c}_{j,s,t}$. On the right-hand-side, the $e^{g_y(1-\sigma)}$ terms cancel out.

$$\left(\hat{w}_t e_{j,s} - \frac{\partial\hat{T}^I_{j,s,t}}{\partial n_{j,s,t}} - \frac{\partial\hat{T}^P_{j,s,t}}{\partial n_{j,s,t}}\right)(\hat{c}_{j,s,t})^{-\sigma}\left(\frac{1}{1+\tau^c_{s,t}}\right) = e^{g_y(1-\sigma)}\chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}$$
$$\forall j,t, \quad \text{and} \quad E+1\leq s\leq E+S \tag{42}$$

We stationarise the Euler equations for savings (8) and (9) by dividing both sides of the respective equations by $e^{-\sigma g_y t}$. On the right-hand-side of the equation, we then need to multiply and divide both terms by $e^{-\sigma g_y(t+1)}$, which leaves a multiplicative coefficient $e^{-\sigma g_y}$.

$$(\hat{c}_{j,s,t})^{-\sigma}\left(\frac{1}{1+\tau^c_{s,t}}\right) = e^{-\sigma g_y}\Bigg[\chi^b_j\rho_s(\hat{b}_{j,s+1,t+1})^{-\sigma} + \beta\big(1-\rho_s\big)(\hat{c}_{j,s+1,t+1})^{-\sigma}\left(\frac{1}{1+\tau^c_{s+1,t+1}}\right)$$
$$\left(1+\hat{r}_{t+1} - \frac{\partial\hat{T}^I_{j,s+1,t+1}}{\partial\hat{b}_{j,s+1,t+1}}\right)\Bigg] \tag{43}$$
$$\forall j,t, \quad \text{and} \quad E+1\leq s\leq E+S-1$$

$$(\hat{c}_{j,E+S,t})^{-\sigma} = e^{-\sigma g_y}\chi^b_j(\hat{b}_{j,E+S+1,t+1})^{-\sigma} \quad \forall j,t \quad \text{and} \quad s=E+S \tag{44}$$

2.5.2 Stationarised Firm Equations

The nonstationary production function (12) can be stationarised by dividing both sides by $e^{g_y t}\tilde{N}$. This stationarises output $\hat{Y}_t$ on the left-hand-side. Because the Cobb-Douglas production function is homogeneous of degree 1, $F(xK,xL)=xF(K,L)$, which means the right-hand-side of the production function is stationarised by dividing by $e^{g_y t}\tilde{N}_t$.

$$\hat{Y}_t = F(\hat{K}_t,\hat{L}_t) \equiv Y_t = Z_t(K_t)^{\gamma}(L_t)^{1-\gamma} \quad \forall t \tag{45}$$

Notice that the growth term multiplied by the labour input drops out in this stationarised version of the production function. We stationarise the nonstationary profit function (14) in the same way, by dividing both sides by $e^{g_y t}\tilde{N}_t$.

$$\hat{PR}_t = F(\hat{K}_t,\hat{L}_t) - \hat{w}_t\hat{L}_t - \big(r_t+\delta\big)\hat{K}_t \quad \forall t \tag{46}$$

The firms' first order equation for labour demand (15) is stationarised by dividing both sides by $e^{g_y t}$. This stationarises the wage $\hat{w}_t$ on the left-hand-side and cancels out the $e^{g_y t}$ term in front of the right-hand-side. To complete the stationarisation, we multiply and divide the $\frac{Y_t}{e^{g_y t}L_t}$ term on the right-hand-side by $\tilde{N}_t$.

$$\hat{w}_t = (1-\gamma)\frac{\hat{Y}_t}{\hat{L}_t} \quad \forall t \tag{47}$$

It can be seen from the firms' first order equation for capital demand (16) that the interest rate is already stationary. If we multiply and divide the $\frac{Y_t}{K_t}$ term on the right-hand-side by $e^{t_y t}\tilde{N}_t$, those two aggregate variables become stationary. In other words, Y_t and K_t grow at the same rate and $\frac{Y_t}{K_t} = \frac{\hat{Y}_t}{\hat{K}_t}$.

$$\begin{aligned} r_t &= \gamma\frac{\hat{Y}_t}{\hat{K}_t} - \delta \quad \forall t \\ &= \gamma\frac{Y_t}{K_t} - \delta \quad \forall t \end{aligned} \tag{16}$$

Equations (45), (47), and 16 imply the following convenient formula for stationarised wage being a function only of the stationary interest rate and parameters:

$$\hat{w}_t = (1-\gamma)\,Z\left(\frac{\gamma Z}{r_t+\delta}\right)^{\frac{\gamma}{1-\gamma}} \quad \forall t \tag{48}$$

2.5.3 Stationarised Market Clearing Equations

The labour market clearing equation (36) is stationarised by dividing both sides by $\tilde{N}_t$.

$$\hat{L}_t = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\hat{\omega}_{s,t}\lambda_j e_{j,s} n_{j,s,t} \quad \forall t \tag{49}$$

The capital market clearing equation (37) is stationarised by dividing both sides by $e^{g_y t}\tilde{N}_t$. Because the right-hand-side has population levels from the previous period $\omega_{s,t-1}$, we have to multiply and divide both terms inside the parentheses by $\tilde{N}_{t-1}$ which leaves us with the term in front of $\frac{1}{1+\tilde{g}_{n,t}}$.

$$\hat{K}_t = \frac{1}{1+\tilde{g}_{n,t}}\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\Big(\hat{\omega}_{s-1,t-1}\lambda_j\hat{b}_{j,s,t} + i_s\hat{\omega}_{s,t-1}\lambda_j\hat{b}_{j,s,t}\Big) \quad \forall t \tag{50}$$

We stationarise the goods market clearing (38) condition by dividing both sides by $e^{g_y t}\tilde{N}_t$. On the right-hand-side, we must multiply and divide the K_{t+1} term by $e^{g_y(t+1)}\tilde{N}_{t+1}$ leaving the coefficient $e^{g_y}(1+\tilde{g}_{n,t+1})$. And the term that subtracts the sum of imports of next period's immigrant savings we must multiply and divide by $e^{g(t+1)}$, which leaves the term e^{g_y}.

$$\hat{Y}_t = \hat{C}_t + e^{g_y}(1+\tilde{g}_{n,t+1})\hat{K}_{t+1} - e^{g_y}\left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J} i_s\hat{\omega}_{s,t}\lambda_j\hat{b}_{j,s,t+1}\right) - (1-\delta)\hat{K}_t \quad \forall t$$
$$\text{where} \quad \hat{C}_t \equiv \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\hat{\omega}_{s,t}\lambda_j\hat{c}_{j,s,t} \tag{51}$$

We stationarise the law of motion for total bequests BQ_t in (39) by dividing both sides by $e^{g_y t}\tilde{N}_t$. Because the population levels in the summation are from period $t-1$, we must multiply and divide the summed term by $\tilde{N}_{t-1}$ leaving the term in the denominator of $1+\tilde{g}_{n,t}$.

$$\hat{BQ}_t = \left(\frac{1+r_t}{1+\tilde{g}_{n,t}}\right)\left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\rho_{s-1}\lambda_j\hat{\omega}_{s-1,t-1}\hat{b}_{j,s,t}\right) \quad \forall t \tag{52}$$

The law of motion for total transfers TR_t in (40) is stationarised by dividing both sides by $e^{g_y t}\tilde{N}_t$.

$$\hat{TR}_t = \left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\lambda_j\hat{\omega}_{s,t}\hat{T}^I_{j,s,t} + \sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\lambda_j\hat{\omega}_{s,t}\hat{T}^C_{j,s,t}\right) \quad \forall t. \tag{53}$$

3 Calibration

3.1 Demographics

We start the section on the `EDGE-M3` calibration with a description of the demographics of the model. Nishiyama (2015) and DeBacker et al. (2019) have recently shown that demographic dynamics are likely the biggest influence on macroeconomic time series, exhibiting more influence than fiscal variables or household preference parameters.

In this section, we characterise the equations and parameters that govern the transition dynamics of the population distribution by age. In `EDGE-M3` , we use Eurostat's projections of mortality rates, fertility rates and net immigration rates.[11]

We define $\omega_{s,t}$ as the number of households of age s alive at time t. A measure $\omega_{1,t}$ of households is born in each period t and live for up to $E+S$ periods, with $S \geq 4$.[12] Households are termed "youth", and do not participate in market activity during ages $1 \leq s \leq E$. The households enter the workforce and economy in period $E+1$ and remain in the workforce until they unexpectedly die or live until age $s = E+S$. We model the population with households age $s \leq E$ outside of the workforce and economy in order most closely match the empirical population dynamics.

[11] Eurostat database: Population and social conditions - Population projections (proj) - Population projections at national level (2015-2080) (proj_15n) http://ec.europa.eu/eurostat/data/database, access 30/08/2018.

[12] Theoretically, the model works without loss of generality for $S \geq 3$. However, because we are calibrating the ages outside of the economy to be one-fourth of S (e.g., ages 21 to 100 in the economy, and ages 1 to 20 outside of the economy), it is convenient for S to be at least 4.

The population of agents of each age in each period $\omega_{s,t}$ evolves according to the following function,

$$\begin{aligned} \omega_{1,t+1} &= \sum_{s=1}^{E+S} f_{s,t}\omega_{s,t} + i_{1,t}\omega_{1,t} \quad \forall t \\ \omega_{s+1,t+1} &= (1-\rho_{s,t})\omega_{s,t} + i_{s+1,t}\omega_{s+1,t} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1 \end{aligned} \tag{54}$$

where $f_s \geq 0$ is an age-specific fertility rate, $i_{s,t}$ is an age-specific net immigration rate, $\rho_{s,t}$ is an age-specific mortality hazard rate.[13] The total population in the economy N_t at any period is simply the sum of households in the economy, the population growth rate in any period t from the previous period $t-1$ is $g_{n,t}$, $\tilde{N}_t$ is the working age population, and $\tilde{g}_{n,t}$ is the working age population growth rate in any period t from the previous period $t-1$.

$$N_t \equiv \sum_{s=1}^{E+S} \omega_{s,t} \quad \forall t \tag{55}$$

$$g_{n,t+1} \equiv \frac{N_{t+1}}{N_t} - 1 \quad \forall t \tag{56}$$

$$\tilde{N}_t \equiv \sum_{s=E+1}^{E+S} \omega_{s,t} \quad \forall t \tag{57}$$

$$\tilde{g}_{n,t+1} \equiv \frac{\tilde{N}_{t+1}}{\tilde{N}_t} - 1 \quad \forall t \tag{58}$$

We discuss the approach to estimating fertility rates $f_{s,t}$, mortality rates $\rho_{s,t}$, and immigration rates $i_{s,t}$ in Sections 3.1.1, 3.1.2, and 3.1.3.

3.1.1 Fertility rates

In `EDGE-M3`, we use Eurostat's baseline projections for fertility rates.[14] Annual data are used until 2070, after which the fertility rates are assumed constant at the 2070 rates. Figure 3 shows the fertility-rate data and the estimated average fertility rates for $E+S=100$ for selected years.

The blue line in Figure 3 shows the 2015 Italian fertility rate per person by age (showing the peak in fertility at age 33). Eurostat baseline projections show modest increases in fertility rates over time, with the values for 2040 and 2070 shown in Figure 3.

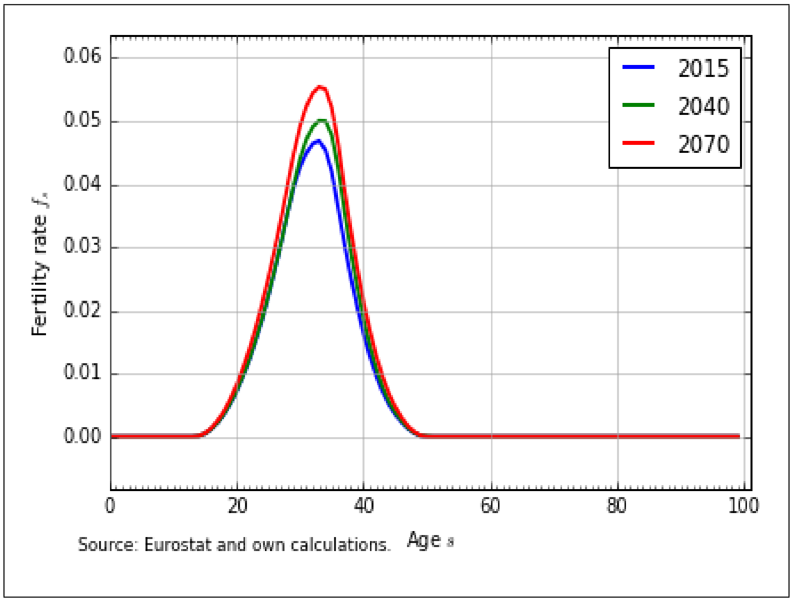
3.1.2 Mortality rates

The mortality rates in `EDGE-M3`, $\rho_{s,t}$, are a one-period hazard rate and represent the probability of dying within one year, given that a household is alive at the beginning of period

[13] The parameter $\rho_{s,t}$ is the probability that a household of age s dies before age $s+1$.

[14] Eurostat database proj_15naasfr. We convert Eurostat fertility per woman data to fertility per person using Eurostat baseline projections for female population compared with total population - Eurostat database proj_15npms. Note that Eurostat fertility data are for live births.

Figure 3: Fertility rates by age ($f_{s,t}$) for $E + S = 100$ selected years



s. We use Eurostat's baseline projections for Italian mortality rates by age.[15] Annual data are used until 2070, after which the mortality rates are assumed constant at the 2070 rates. Figure 4 shows the mortality rate data and the corresponding model-period mortality rates for $E + S = 100$. We constrain the mortality rate to be 1.0 or 100 percent at the maximum age of 100.

3.1.3 Immigration rates

`EDGE-M3` uses net immigration rates from Eurostat.[16] Annual data are used until 2070, after which the net immigration rates are assumed constant at the 2070 rates. Figure 5 shows the net immigration rates for selected years, showing that the general pattern of immigration by age is projected to continue, with the rates rising from 2015 to 2040 (the peak year is 2039), before falling gradually until 2070.

At the end of Section 3.1.4, we describe a small adjustment that we make to the immigration rates after a large number of periods in order to make computation of the transition path equilibrium of the model compute more robustly.

3.1.4 Population steady-state and transition path

This model requires information about mortality rates $\rho_{s,t}$ in order to solve for the household's problem each period. It also requires the net immigrations rates $i_{s,t}$ for the external balance.

[15]Eurostat database proj_15naasmr. As the mortality data is provided separately for male and female, we calculate the mortality per person using Eurostat baseline projections for male and female population compared with total population - Eurostat database proj_15npms.

[16]Eurostat database proj_15nanmig. As the data are in levels of net immigration, we calculate the rates using the Eurostat baseline projections for total population - Eurostat database proj_15npms.

Figure 4: Mortality rates by age ($\rho_{s,t}$) for $E+S=100$ selected years

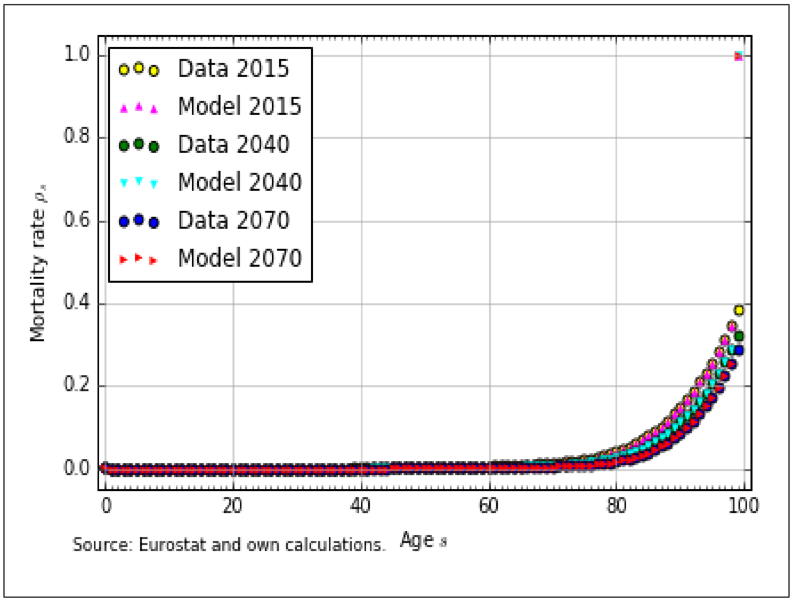
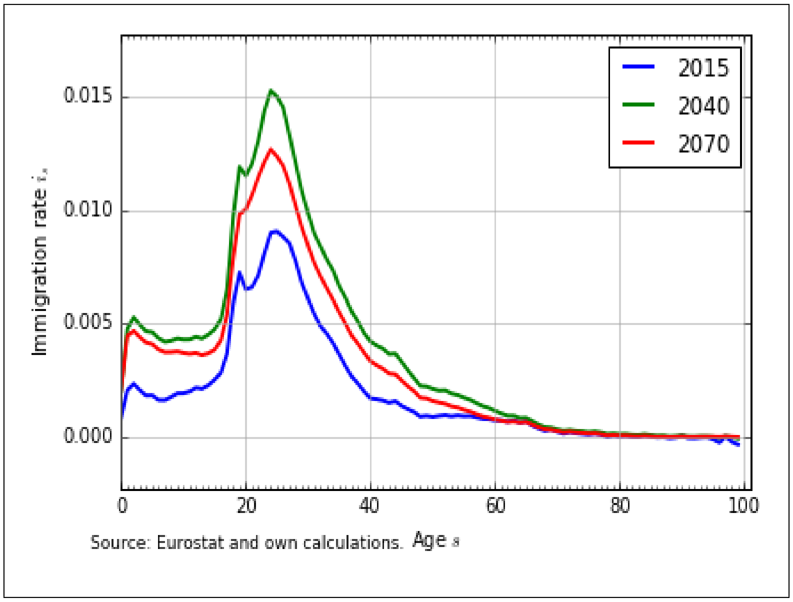


Figure 5: Immigration rates by age ($i_{s,t}$) for $E+S=100$ selected years



Additionally, the steady-state stationary population distribution $\bar{\omega}_s$ and population growth rate $\bar{g}_n$ are needed as well as the full transition path of the stationary population distribution $\hat{\omega}_{s,t}$ and population grow rate $\tilde{g}_{n,t}$ from the current state to the steady-state. To solve for the steady-state and the transition path of the stationary population distribution, we write the stationary population dynamic equations (59) and their matrix representation (60).

$$
\begin{aligned}
\hat{\omega}_{1,t+1} &= \frac{(1-\rho_{0,t})\sum_{s=1}^{E+S} f_{s,t}\hat{\omega}_{s,t} + i_{1,t}\hat{\omega}_{1,t}}{1+\tilde{g}_{n,t+1}} \quad \forall t \\
\hat{\omega}_{s+1,t+1} &= \frac{(1-\rho_{s,t})\hat{\omega}_{s,t} + i_{s+1,t}\hat{\omega}_{s+1,t}}{1+\tilde{g}_{n,t+1}} \qquad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1
\end{aligned} \tag{59}
$$

$$
\begin{bmatrix} \hat{\omega}_{1,t+1} \\ \hat{\omega}_{2,t+1} \\ \hat{\omega}_{2,t+1} \\ \vdots \\ \hat{\omega}_{E+S-1,t+1} \\ \hat{\omega}_{E+S,t+1} \end{bmatrix} = \frac{1}{1+g_{n,t+1}} \times ...
$$

$$
\begin{bmatrix}
(1-\rho_{0,t})f_{1,t}+i_{1,t} & (1-\rho_{0,t})f_{2,t} & (1-\rho_{0,t})f_{3,t} & \dots & (1-\rho_{0,t})f_{E+S-1,t} & (1-\rho_{0,t})f_{E+S,t} \\
1-\rho_{1,t} & i_{2,t} & 0 & \dots & 0 & 0 \\
0 & 1-\rho_{2,t} & i_{3,t} & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \dots & i_{E+S-1,t} & 0 \\
0 & 0 & 0 & \dots & 1-\rho_{E+S-1,t} & i_{E+S,t}
\end{bmatrix}
\begin{bmatrix} \hat{\omega}_{1,t} \\ \hat{\omega}_{2,t} \\ \hat{\omega}_{2,t} \\ \vdots \\ \hat{\omega}_{E+S-1,t} \\ \hat{\omega}_{E+S,t} \end{bmatrix} \tag{60}
$$

We can write system (60) more simply in the following way.

$$
\boldsymbol{\hat{\omega}}_{t+1} = \frac{1}{1+g_{n,t+1}}\boldsymbol{\Omega}\boldsymbol{\hat{\omega}}_t \quad \forall t \tag{61}
$$

The stationary steady-state population distribution $\boldsymbol{\bar{\omega}}$ is the eigenvector $\boldsymbol{\omega}$ with eigenvalue $(1+\bar{g}_n)$ of the matrix $\boldsymbol{\Omega}$ that satisfies the following version of (61).

$$
(1+\bar{g}_n)\boldsymbol{\bar{\omega}} = \boldsymbol{\Omega}\boldsymbol{\bar{\omega}} \tag{62}
$$

Proposition 1. If the age $s = 1$ immigration rate is $i_{1,t} > -(1-\rho_{0,t})f_{1,t}$ and the other immigration rates are strictly positive $i_{s,t} > 0$ for all $s \geq 2$ such that all elements of $\boldsymbol{\Omega}$ are nonnegative, then there exists a unique positive real eigenvector $\boldsymbol{\bar{\omega}}$ of the matrix $\boldsymbol{\Omega}$, and it is a stable equilibrium.

Proof. First, note that the matrix $\boldsymbol{\Omega}$ is square and non-negative. This is enough for a general version of the Perron-Frobenius Theorem to state that a positive real eigenvector exists with a positive real eigenvalue. This is not yet enough for uniqueness. For it to be unique by a version of the Perron-Frobenius Theorem, we need to know that the matrix is irreducible.

This can be easily shown. The matrix is of the form

$$\mathbf{\Omega} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & 0 & \dots & 0 & 0 & 0 \\ 0 & * & * & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & 0 \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}$$

Where each * is strictly positive. It is clear to see that taking powers of the matrix causes the sub-diagonal positive elements to be moved down a row and another row of positive entries is added at the top. None of these go to zero since the elements were all non-negative to begin with.

$$\mathbf{\Omega}^2 = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ 0 & * & * & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & 0 \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix} ; \quad \mathbf{\Omega}^{S+E-1} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}$$

$$\mathbf{\Omega}^{S+E} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \end{bmatrix}$$

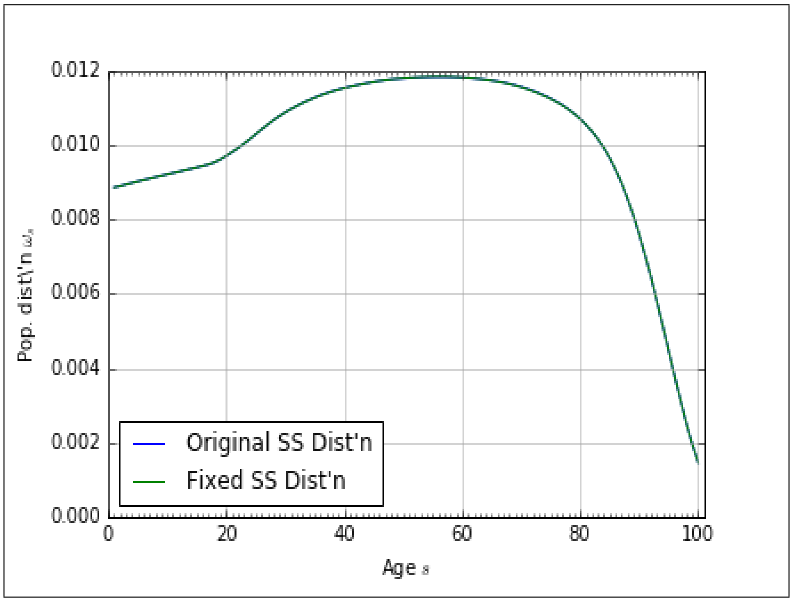
Existence of an $m \in \mathbb{N}$ such that $\left(\mathbf{\Omega^m}\right)_{ij} \neq 0 \;\; (> 0)$ is one of the definitions of an irreducible (primitive) matrix. It is equivalent to saying that the directed graph associated with the matrix is strongly connected. Now the Perron-Frobenius Theorem for irreducible matrices gives us that the equilibrium vector is unique.

We also know from that theorem that the eigenvalue associated with the positive real eigenvector will be real and positive. This eigenvalue, p, is the Perron eigenvalue and it is the steady state population growth rate of the model. By the PF Theorem for irreducible matrices, $|\lambda_i| \leq p$ for all eigenvalues λ_i and there will be exactly h eigenvalues that are equal, where h is the period of the matrix. Since our matrix $\mathbf{\Omega}$ is aperiodic, the steady state growth rate is the unique largest eigenvalue in magnitude. This implies that almost all initial vectors will converge to this eigenvector under iteration. □

For a full treatment and proof of the Perron-Frobenius Theorem, see Suzumura (1983). Because the population growth process is exogenous to the model, we calibrate it to annual age data for age years $s = 1$ to $s = 100$.

Figure 6 shows the steady-state population distribution $\bar{\boldsymbol{\omega}}$ and the population distribution after 240 periods $\hat{\boldsymbol{\omega}}_{240}$. Although the two distributions look very close to each other, they are not exactly the same.

Figure 6: Theoretical steady-state population distribution vs. population distribution at period $t = 120$



Further, we find that the maximum absolute difference between the population levels $\hat{\omega}_{s,t}$ and $\hat{\omega}_{s,t+1}$ was 1.72×10^{-6} after 320 periods. That is to say, that after 320 periods, given the estimated mortality, fertility, and immigration rates, the population has not achieved its steady state. For convergence in our solution method over a reasonable time horizon, we want the population to reach a stationary distribution after T_1 periods. To do this, we artificially impose that the population distribution in period $t = T_1 = 320$ $(4S)$ is the population steady-state. As can be seen from Figure 6, this assumption is not very restrictive. Figure 7 shows the change in immigration rates that would make the period $t = T_1 = 320$ population distribution equal be the steady-state. The maximum absolute difference between any two corresponding immigration rates in Figure 7 is 0.00022.

The most recent year of population data come from Eurostat 2015 population estimates.[17] We use those data and the population transition matrix (61) to generate the transition path of the population distribution over the time period of the model. Figure 8 shows the progression from the 2015 population data to the fixed steady-state at period $t = 320$. The time path of the growth rate of the economically active population $\tilde{g}_{n,t}$ is shown in Figure 9.

[17] Eurostat database proj_15npms.

Figure 7: Original immigration rates vs. adjusted immigration rates to make fixed steady-state population distribution

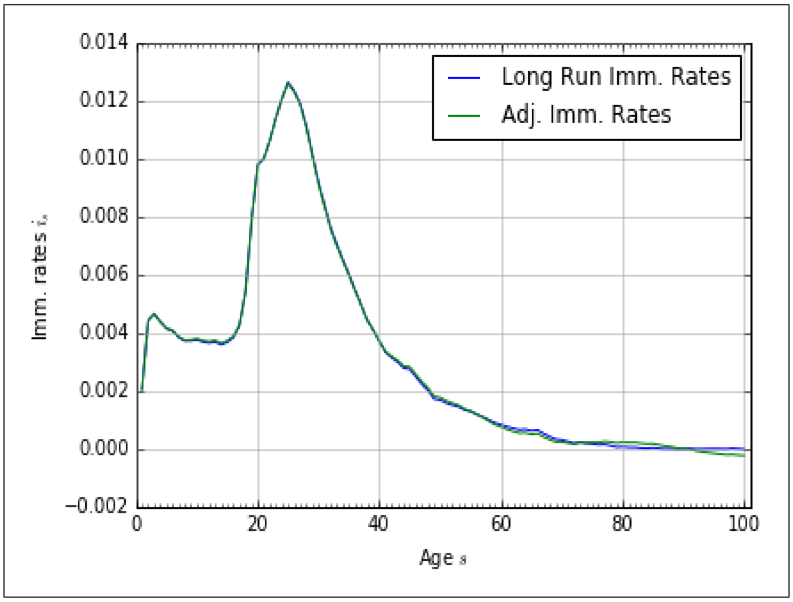


Figure 8: Stationary population distribution at periods along transition path

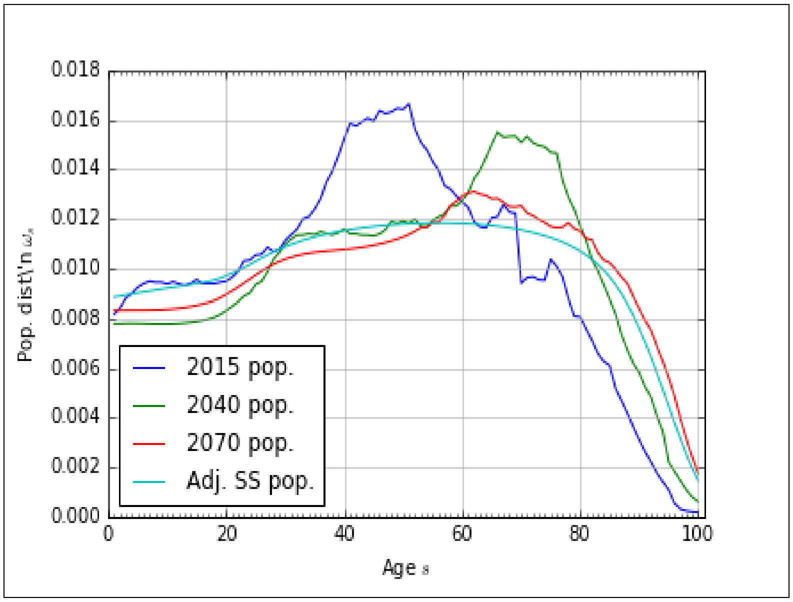
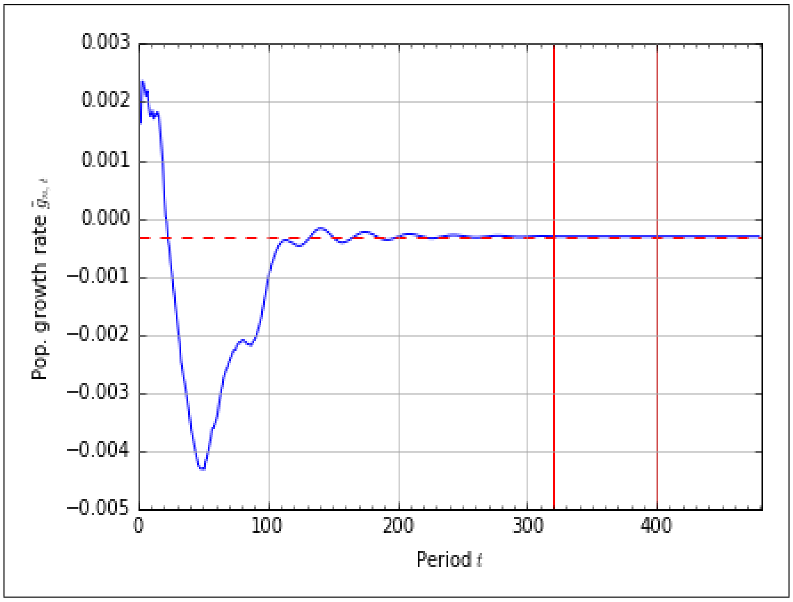


Figure 9: Time path of the population growth rate $\tilde{g}_{n,t}$



3.2 Lifetime Earnings Profiles

Among households in `EDGE-M3`, we model both age heterogeneity and within-age ability heterogeneity. We use this heterogeneity in ability (or equivalently, in productivity) to generate the income heterogeneity that we see in the data.

Differences among workers' productivity in terms of intrinsic ability is one of the key dimensions of heterogeneity to model in a micro-founded macroeconomy. In this section, we characterise this heterogeneity as deterministic lifetime productivity paths to which new cohorts of agents in the model are randomly assigned. In `EDGE-M3`, households' labour income comes from the equilibrium wage and the agent's endogenous quantity of labour supply. In this section, we augment the labour income expression with an individual productivity $e_{j,s}$, where j is the index of the ability type or path of the individual and s is the age of the individual with that ability path.

$$\text{labour income:} \quad x_{j,s,t} \equiv w_t e_{j,s} n_{j,s,t} \quad \forall j,t \quad \text{and} \quad E+1 \leq s \leq E+S \tag{63}$$

In this specification, w_t is an equilibrium wage representing a portion of labour income that is common to all workers. Individual quantity of labour supply is $n_{j,s,t}$, and $e_{j,s}$ represents a labour productivity factor that augments or diminishes the productivity of a worker's labour supply relative to average productivity.

We calibrate deterministic ability paths such that each income group is defined in terms of lifetime income and has a different life-cycle profile of earnings. The distribution of income and wealth are often focal components of macroeconomic models. As such, we use a calibration of 7 deterministic lifetime ability paths representing the bottom 25 percentiles, then the next 25, 20, 10, 10 and 9 percentiles, with the highest ability group representing the top percentile. Alvaredo and Pisano (2010) show that the income attributable to the top earners has grown in importance in recent decades (though to a lesser extent than in Anglo-Saxon countries). The latter observation drove our choice to divide abilities in the way described, although nothing would prevent in principle to choose a different representation of ability paths, for instance assuming equally-sized quantiles. In the latter case the methodology used for calibration would not change, therefore hereafter we only detail the case for 7 ability groups.

We employ survey data from Banca d'Italia in our calibration of these life cycle profiles of earnings for Italy. The data set is the "Archivio storico dellIndagine sui bilanci delle famiglie italiane, 1977-2014". From this dataset we used the files labelled COMP and LDIP, vesrion 9.1 (July 2017).[18] The extracted data, after removing observations that are either outside of the age interval 20-80, with missing values or obtaining a hourly wage smaller than EUR 0.50, are summarised in Table 3. The data mostly comprise full-time workers (about 91% of the sample) with an average amount of working hours per week of 37.7.

Table 3: Descriptive Statistics for the data used to estimate lifetime earnings profiles

Variable	No. Obs.	Mean	Std. Dev.	Other Information
Age	83,594	40.6972	10.9868	Range from 20 to 80
Yearly income	83,594	13,207	7,562	
Working months	83,594	11.3841	2.0032	
Hours worked (weekly)	83,594	37.7460	8.6970	
Part-time work?	83,594	.0907	.2872	Binary variable (1=Yes)

The Banca d'Italia data does not provide hourly wages. To derive them we further exploit the following information: number of worked months in a year, yearly earnings from labour income (including non-monetary benefits), number of hours worked on average in a working week. We consider both full and part-time workers, and both employed and self-employed workers. We assume 46 out of 52 weeks worked for a full-time worker in a year. The nominal wages are then converted to real wages using annual consumer price indexes (CPIs),[19] expressed relative to the reference year (2010=100). Then, our measure of hourly real wage W is obtained as:

$$W = \frac{yearlyearnings}{worked\_hours \times worked\_months \times \frac{46}{52} \times \frac{CPI}{100}} \quad (64)$$

Since hourly earnings vary over the lifecycle, we wish to assign individuals into one of

[18]Data and the related documentation were downloaded in January 2017, in Stata format, from the URL: https://www.bancaditalia.it/statistiche/tematiche/indagini-famiglie-imprese/bilanci-famiglie/index.html?com.dotmarketing.htmlpage.language=1

[19]We employed consumer price indexes (CPIs) downloaded from the St. Luis FRED (https://fred.stlouisfed.org). The series includes annual CPIs, expressed relative to the reference year 2010=100, and is not seasonally adjusted.

our J earnings ability types based on their potential earnings over their lifetime. As we do not observe earnings over a full lifetime for households in our data, we use the following methodology to identify ability groups based on lifetime income. First, we divide the data into six age brackets: 20 to 30, 31 to 40, 41 to 50, 51 to 60, 61 to 70, and 70 to 80-year olds. Within each bracket individuals were ranked according to their mean hourly wage. Individuals are then assigned to ability groups based on their positioning within the assigned bracket: ability group 1 if in the 0-24th percentile, group 2 if in 25-49th, group 3 if 50-69th, group 4 if 70-79th, group 5 if 80-89th, group 6 if 90-99th, and finally group 7 in if the 100th percentile. For each of these 7 ability groups we then obtain a panel dataset by year and individual, where the age and hourly wage of the person is observed. Separately for each ability group, we run panel fixed-effects regressions to derive the relation between age and hourly wage, according to the following cubic regression model:

$$ln(w_{i,t}) = \alpha_i + \beta_1 age_{i,t} + \beta_2 age_{i,t}^2 + \beta_3 age_{i,t}^3 + \varepsilon_{i,t} \tag{65}$$

The coefficients obtained from estimating this regression model are used to predict new values for each individual in the data at every age, even if the individual was not in the data at that age. Because there are few individuals above age 80 in the data, our data series were further extended up to age 100 by linearly extrapolating new wages, assuming that the hourly wage at age 100 is equal to the hourly wage at 80 divided by 1.5. Finally, in order to have differentiable curves for each ability type, the data was smoothed using a "lowess" function with a 0.3 bandwidth.

Figure 10 shows a calibration for $J = 7$ deterministic lifetime ability paths $e_{j,s}$ corresponding to the following labour income percentile groupings.

$$\lambda_j = [0.25, 0.25, 0.2, 0.1, 0.1, 0.09, 0.01] \tag{66}$$

Our calibration allows for each lifetime income group to have a different life-cycle profile of earnings. This helps us match the distributions of income and wealth observed in the data. Matching these distributions is key aspect of our model and adds relevance to the distributional analyses it provides.

3.3 Calibration of Preference Parameters

Values for many exogenous variables and parameters of `EDGE-M3` come outside the model. However, two sets of parameters, the utility weight on the disutility of labour χ^n_s and the utility weight on bequests χ^b_j, are chosen to match the steady-state values of the model with their real-world counterparts in the Italian economy. We describe the calibration process for these parameters in this section.

3.3.1 Hours Worked Calibration

In order to calibrate the utility weight on the disutility of labour, χ^n_s, we choose the 80 parameter values $\{\chi^n_s\}_{s=E+1}^{E+S}$ to match 80 moments from the data. To do this, we begin with

Figure 10: Exogenous life cycle income ability paths $\log(e_{j,s})$ with $S = 80$ and $J = 7$

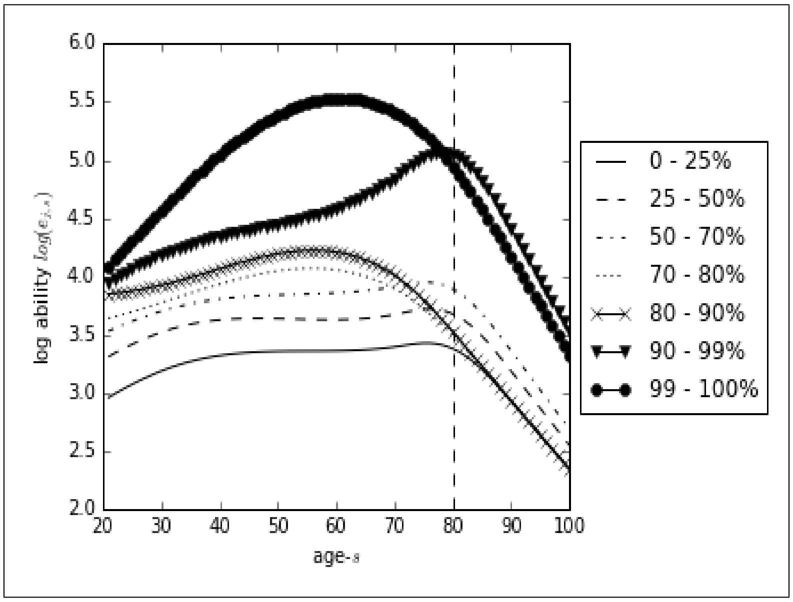


Table 4: Log Wage Regressions, by Lifetime Income Group

Lifetime income groups (percentiles)	Constant	Age	Age^2	Age^3	Observations
0 to 25	1.617***	0.0957***	-0.00174**	0.0000105*	19222
	(5.23)	(4.19)	(-3.17)	(2.45)	
25 to 50	1.932***	0.101***	-0.00195***	0.0000123***	20040
	(9.2)	(6.57)	(-5.26)	(4.22)	
50 to 70	2.515***	0.0729***	-0.00135**	0.00000846**	16027
	(10.09)	(4.08)	(-3.22)	(2.62)	
70 to 80	3.692***	-0.0202	0.00109	-0.0000108	8006
	(6.47)	(-0.49)	(1.14)	(-1.48)	
80 to 90	4.415***	-0.0607	0.00198*	-0.0000170*	8015
	(7.32)	(-1.41)	(1.97)	(-2.22)	
90 to 99	2.510**	0.105	-0.00207	0.0000149	8014
	(2.98)	(1.76)	(-1.5)	(1.43)	
99 to 100	3.000	0.0430	0.000647	-0.0000109	805
	(0.44)	(0.08)	(0.05)	(-0.11)	

Source: Own calculations using Banca d'Italia data.
t statistics in parentheses
* Significant at the 5 percent level ($p < 0.05$).
** Significant at the 1 percent level ($p < 0.01$).
*** Significant at the 0.1 percent level ($p < 0.001$).

the first order condition on the households choice of labour supply (7) rewritten as:

$$\tilde{w}\left(1-\tilde{\tau_s}^{l}-\tilde{\tau_s}^{p}\right)(\tilde{c}_s)^{-\sigma}\left(\frac{1}{1+\tilde{\tau_s^c}}\right)=\tilde{\chi}_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{\tilde{n}_s}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\tilde{n}_s}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}},\quad \forall s \tag{67}$$

where $\tilde{w}$ is gross wage including social insurance contributions (in thousands EUR), $\tilde{\tau_s}^{l}$ is labour income direct tax rate, $\tilde{\tau_s}^{p}$ is payroll tax rate, $\tilde{\tau_s^c}$ is consumption tax rate, $\tilde{c}_s$ is consumption of individual of age s (in thousands EUR), $\frac{\tilde{n}_s}{\tilde{l}}$ is hours worked as a percentage of total time endowment of individual of age s, σ is a coefficient of risk aversion (the inverse of the intertemporal elasticity of substitution), $\tilde{\chi}_s^n$ is a scale parameter that influences the relative disutility of labour to the utility of consumption. $b>0$ is a scale parameter and $\upsilon>0$ is a curvature parameter of an elliptical utility function for labour as described in Evans and Phillips (2018). The values for b and υ are found by matching the marginal utility from the elliptical utility function to the marginal utility from a constant Frisch elasticity utility function, where the Frisch elasticity is calibrated based on econometric studies of the Frisch elasticity.

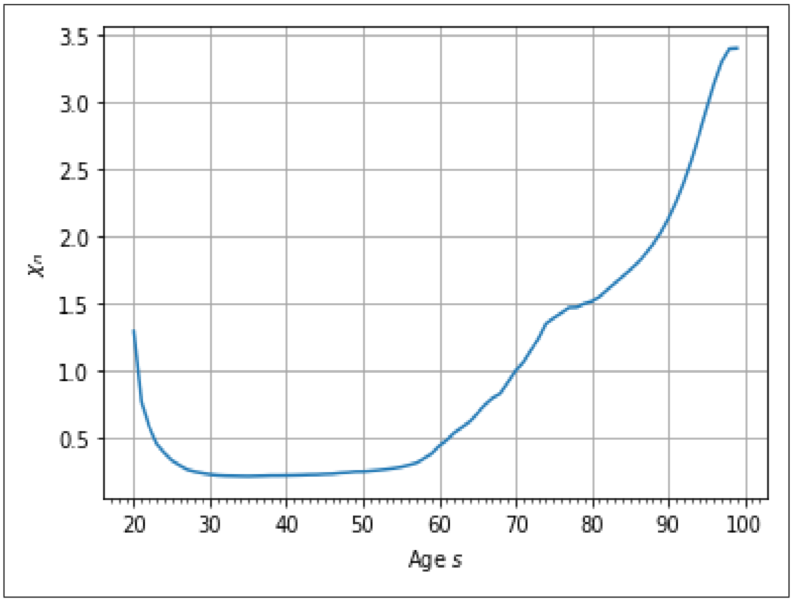
In order to identify $\tilde{\chi}_s^n$ we have to reformulate the household first order condition (67) as:

$$\tilde{\chi}_s^n=\frac{\tilde{w}\left(1-\tilde{\tau_s}^{l}-\tilde{\tau_s}^{p}\right)(\tilde{c}_s)^{-\sigma}\left(\frac{1}{1+\tilde{\tau_s^c}}\right)}{\left(\frac{b}{\tilde{l}}\right)\left(\frac{\tilde{n}_s}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\tilde{n}_s}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}} \tag{68}$$

To determine the value of $\tilde{\chi}_s^n$ we use data on average gross wage, average consumption by individuals of age s, and data on average hours worked. The data on average gross wage and on average hours worked are the EU-SILC data. Since the EU-SILC data do not contain detailed information on consumption and expenditures, these have been imputed based on the information from the Household Budget Surveys being matched to EU-SILC [20] data through estimated Engel curves. Note that the χ_s^n will be measured in data units that may not directly correspond to the model consumption units χ_s^n is measured in. Wage is defined as gross hourly wage from the discrete choice labour supply model (being one of the extensions to the `EUROMOD` microsimulation model) multiplied by the assumed total time endowment of 4000 hours per year and divided by thousand since we define data units as thousand EUR. We find c_s form the data by averaging over annual consumption for all individuals of age s in our data for years 2008, 2010, 2012, and 2014. Similarly, we find n_s by averaging hours worked over all individuals of age s over the years 2008, 2010, 2012, and 2014. To express annual hours worked as a percentage of total time endowment we assume 4000 hours per year as total time endowment.

[20]EU-SILC abbreviation stands for European Union Statistics on Income and Living Conditions, for more information see http://ec.europa.eu/eurostat/web/microdata/european-union-statistics-on-income-and-living-conditions.

Figure 11: Calibrated values of $\tilde{\chi}^n_s$



In order to account for the differences between the model units (i.e. consumption units) and data units (i.e. thousands of EUR) the scaling factor has been introduced for determining the steady-state variables in the model. The relation between parameter χ^n_s (in model units) and parameter $\tilde{\chi}^n_s$ (in data units) is as follows:

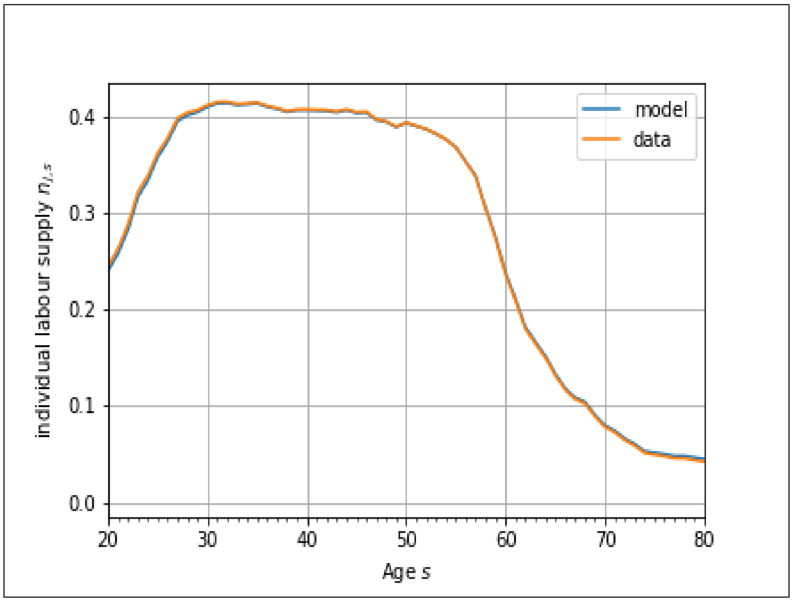
$$\chi^n_s = \tilde{\chi}^n_s factor^{\sigma-1} \tag{69}$$

Thus, by estimating $\tilde{\chi}^n_s$ using the data on wages, consumption, and labour supply, one can determine model parameters up to a scale. That scale is a function of the model scale parameter, $factor$ (for a definition of a $factor$, see Section 2.3.1). Figure 11 shows the 80 calibrated values for $\tilde{\chi}^n_s$ (values for ages 80-99 have been linearly extrapolated).

To match the data on hours worked as close as possible in `EDGE-M3`, an iterative procedure is applied to calibrate $\tilde{\chi}^n_s$. In the first iteration the observed hours worked data $\tilde{n}_s$ are used. Then the model is solved for the steady state several times using each time a steady-state solution for hours worked, n_ss from the previous solution until n_ss is not significantly different from the observed data.

Figure 12 shows how closely the average steady-state labour supply by age from the model matches the average hours by age in the Italian data. In this graph, we only show ages for which we had observations, i.e. until age 80.

Figure 12: Life-cycle average labour supply: model vs. data



3.3.2 Bequests Calibration

This section describes how we calibrate the distribution of total bequests BQ_t to each living household of age s and lifetime income group j. The matrix that governs this distribution $\zeta_{j,s}$ is seen in the household budget constraint (1).

Allowing the χ_j^b scale parameter on the warm glow bequest motive in (1) to vary by lifetime income group is critical for matching the distribution of wealth. Since individuals in the model have no income uncertainty because each lifetime earnings path $e_{j,s}$ is deterministic, model agents thus hold no precautionary savings. Calibrating the χ_j^b for each income group j allows us to recapture in a reduced form way some of the characteristics that individual income risk provides.

Because agents face mortality risk, and because of the functional form for the warm glow bequest motive, they prefer to hold some savings for bequest at each age, in the chance they die before the next period. Including this term is essential to generating the skewed distribution of wealth that exists in the data.

To calibrate χ_j^b we choose seven values $\{\chi_j^b\}_{j=1}^7$ to match nine moments from the data. These moments include the share of wealth held by each of the following percentile groups in the wealth distribution: 0-24% , 25-49%, 50-69%, 70-79%, 80-89%, 90-99%, and 99-100%, the Gini coefficient from the wealth distribution, and the variance of log wealth.[21] This is done by minimising the sum of squared percent deviations between average steady-state wealth values in our baseline model and the corresponding values from the data. These seven calibrated values of χ_j^b are the following: $[4.0, 116.0, 346.0, 410.0, 604.0, 1304.0, 3000.0]$.

Equation (1) highlights how these bequests are distributed to other model agents. The

[21] Our reference data is net worth value in 2015 from the ECB's Household Finance and Consumption Survey (HFCS) data, second wave.

Table 5: Wealth distribution moments: model and data

Moment	Data	Model
Share of wealth held by the 0-24% percentile group	0.006	0.025
Share of wealth held by the 25-49% percentile group	0.093	0.1
Share of wealth held by the 50-69% percentile group	0.170	0.168
Share of wealth held by the 70-79% percentile group	0.126	0.122
Share of wealth held by the 80-89% percentile group	0.176	0.173
Share of wealth held by the 90-99% percentile group	0.312	0.352
Share of wealth held by the 99-100% percentile group	0.117	0.051
Gini coefficient from wealth distribution	0.60	0.56
Variance of log wealth	3.5	2.0

Source: the authors

term BQ_t represents total bequests from individuals who died at the end of period $t-1$. We assume that bequests are distributed evenly across all ages to those in the same lifetime income group. Given available data, it is difficult to precisely calibrate the distribution of bequests from the data, both across income types j and across ages s. We find that this assumption helps to reproduce the empirical distribution of wealth, where wealth is highly concentrated at the top. The next section describes how we determined the bequest share matrix $\zeta_{j,s}$ from equation (1) based on data available for Italy.

3.3.3 Bequest Shares Calibration

Below we describe how we determined the bequest share matrix $\zeta_{j,s}$ in equation (1) using scarce data available for Italy. The following source data were used:

i. average capitalised received transfers for tenths of households' by net wealth in 2002 (Table A5 Wealth and transfers in Cannari and Alessio (2008))

ii. average (total) capitalised received transfers in 2002 (Table 1 Wealth and transfers in Cannari and Alessio (2008))

iii. average (total) capitalised received inheritances in 2002 (Table 1 Wealth and transfers in Cannari and Alessio (2008))

iv. median values of household net wealth by household income quintiles in 2002 and 2014 (Table E2 Median values of household net wealth in Banca d'Italia (2004) and Banca d'Italia (2015))

v. average capitalised received inheritances by six age brackets (up to 30, 31-40, 41-50, 51-60, 61-70 and over 71) in 2002 (Table A1 Intergenerational transfers by age of household head in Cannari and Alessio (2008))

vi. share of households that have received transfers by the six age brackets in 2002 (Table A1 Intergenerational transfers by age of household head in Cannari and Alessio (2008))

vii. median value of net wealth by five age groups (up to 30, 31-40, 41-50, 51-65 and over 65) in 2002 and in 2014 (Table E2 Median values of household net wealth in Banca d'Italia (2004) and Banca d'Italia (2015))

viii. average over ages net worth in the seventh ability group relative to the sixth ability group based on the Finance and Network (2016) data

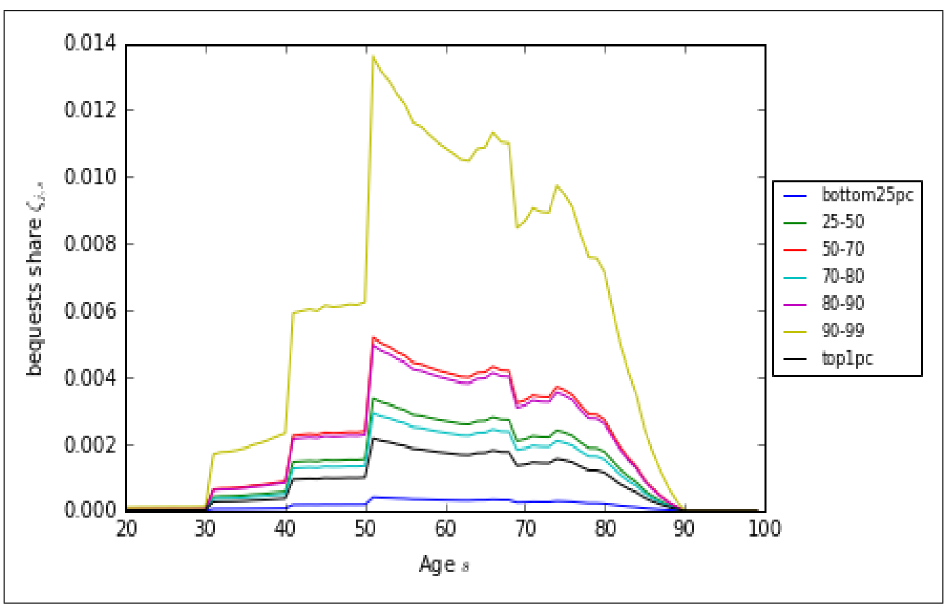
The model base year is 2015. However, the detailed information of bequests distribution by age groups and wealth deciles for Italy was only found for 2002. Thus, values for year 2002 were uprated to year 2014, the closest year to the base year for which the survey results on bequests were published in Banca d'Italia (2015). Below the methodology of calculating the bequest share matrix by model ages (i.e. from 20 to 99) and seven ability groups is described.

To get the average capitalised received inheritances for tenths of households' by net wealth in 2002, the average (total) capitalised received inheritances in 2002 iii) was multiplied by the ratios of i) to ii) for tens of households by net wealth (i.e. the same distribution of inheritances as transfers among households deciles was assumed). Then to estimate the average received inheritances by households' deciles in 2014, the median value of household net wealth by household income quintiles iv) was multiplied by the ratio of the average received inheritances by households' deciles in 2002 to the median value of household net wealth by household income quintiles in 2002. To get the average capitalised received inheritances by the six age groups in 2002, the average (total) capitalised received inheritances in 2002 iii) was multiplied by the the ratios of v) to ii) for the six age brackets (i.e. the same distribution of inheritances as transfers among households age groups was assumed). To calculate the average capitalised received inheritances by the six age groups in 2014, median value of net wealth in 2014 by the six age groups was multiplied by the ratio of the average capitalised received inheritances by the six age groups in 2002 to the median net wealth in 2002 by the six age groups. The average capitalised received inheritances by the six age groups in 2002 were weighted by the shares of the households that had received transfers vi). In the next step to get the matrix of received bequests with age and ability type dimensions for 2014, the average capitalised received inheritances for each age group were multiplied by the share of the average received inheritances by each net worth decile in total received inheritances. In this way, a matrix of inheritances by the six age groups and the ten net worth deciles in 2014 was obtained.

In the model there are generations modelled by yearly ages instead of six age groups and seven ability groups instead of deciles. Thus, the matrix of the average received bequests by the six age groups and the ten net worth deciles in 2014 need to be calculated for yearly ages 20-99 and seven ability groups. First, since using only information by deciles we cannot determine the seventh ability group (i.e. the richest 1% individuals), we would need to use additional information to split the sixth ability group into the sixth and the seventh, in line with the model structure. First, the matrix by the six ages and the six ability types was determined. To calculate the values for the first ability type in line with the model (i.e. the first 25. percentile), for each age group we sum a 0.4 of the first decile value, a 0.4 of the second decile value and a 0.2 of the third decile value. To calculate the values for the second ability type (i.e. the next 25. percentile), for each age group we sum a 0.2 of the third decile value, 0.4 of the fourth decile value and a 0.4 of the fifth decile value. To calculate the values for the third ability type (i.e. the next 20. percentile), for each age group we sum a 0.5 of the sixth decile value and a 0.5 of the seventh decile value. Finally, the fourth, the fifth and the sixth ability values were set equal to the corresponding eighth, ninth and tenth values of the deciles matrix. Next, to split the sixth ability group into the sixth and the seventh ability types, we assumed that the ratio of the averaged over ages bequests value received by the seventh ability group relative to the averaged over ages bequests value received by the sixth ability group was the same as for net worth in the line with the HFCS data for Italy (Finance and Network (2016)). A constraint that the weighted average of the new sixth ability and the seventh ability values must be equal to the previous sixth ability values was met.

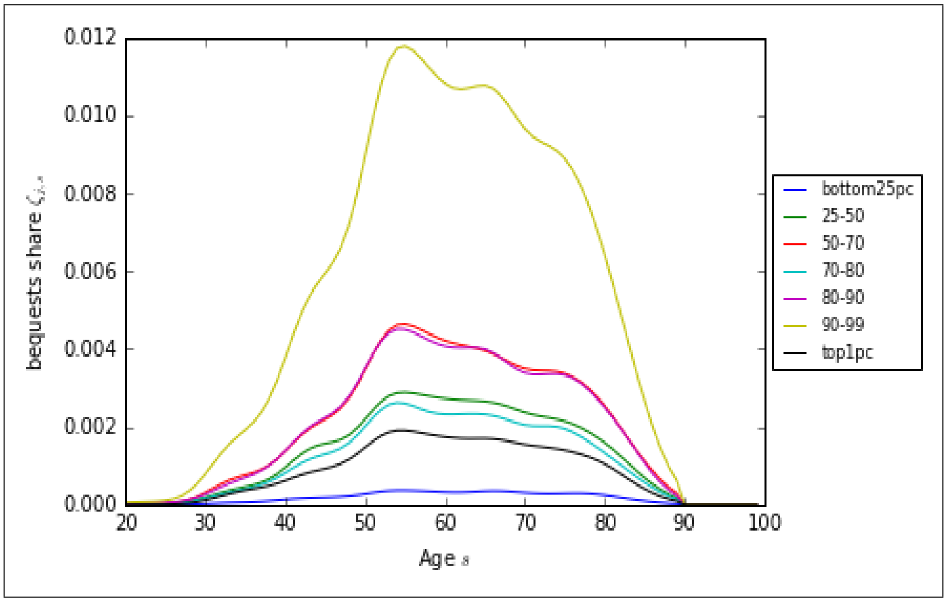
In the next step, the values for the six age groups have to be split into values for the single (eighty) ages. To this end, the age shares of population in total population was used. As a result, a matrix of average bequests received by ages 20-99 and the seven ability types was obtained. Finally, the bequest shares matrix was obtained by calculating the shares of the values of the average bequests by the single ages and the seven ability groups (scaled by the ability weights) in the sum of all bequests across ages and ability types (see Figure 13). Since without any correction the elderly people 99-year olds would receive much greater bequests over time as compared with the starting period (base year) leading to an odd consumption-savings profiles, the values for 81-89-year olds have been lowered using a factor of 0.9, 0.8,...,0.1, correspondingly for these nine ages. In addition, it was assumed that individuals at age 90 or more do not receive bequests.

Figure 13: Bequest shares by age (s) and ability group (j) - data



Finally, we need to smooth the data since they are stepped due to data being only available for ages groups (mostly decades). To this end we apply the Multivariate Kernel Density Estimator (MVKDE) with the bandwidth parameter equal to 0.2. This parameter value is quite low since we do not want to oversmooth. In the model the smoothed matrix is used (see Figure 14).

Figure 14: Bequest shares by age (s) and ability group (j) - smoothed values



3.4 Calibration of Tax Functions with Microsimulation Data

3.4.1 Income Tax Functions

The model calibration requires to represent average and marginal tax rates on income. In this section we illustrate the methodology used to estimate taxation of labour and capital income, while in the following section 3.4.2 we discuss social insurance contributions levied on labour income.

In order to estimate income tax functions $T^I_{s,t}(x, y)$ (refer again to previous Section 2.3.1) the algorithm requires the following microdata: labour income; capital income; effective tax rates on total income; marginal tax rates, separately for labour and capital income. The microdata we use are at the individual level (we believe individual data better capture the characteristics of the Italian tax system compared to household-level data) and come from the `EUROMOD` model. `EUROMOD` employs survey data from EU-SILC (EU Statistics on Income and Living Conditions) and we use the most recent available wave at the time of writing (as stated already, this is for the year 2015).

We defined labour income as earned income, which is the sum of wages, salaries and self-employment income. Capital income was defined as the sum of income from investment, pension and property. We consequently obtain labour income summing up `EUROMOD`'s variables *yem* (wage employment income) and *yse* (self employment income), capital income summing up *ypp* (private pension), *yiy* (investment income) and *ypr* (property income), labour taxes summing up *tinna_s* and *tinrg_s*, and capital taxes summing up *tinktcp_s*, *tinktdt_s*, *tinktdv_s*, *tinktbd_s*, *tinktgb_s*, *tprmb_s*, *tprob_s* and *tinrt_s* (all the latter variables starting with the letter t are for taxes and are endogenously computed by the `EUROMOD` model). `EUROMOD` also provides a functionality to compute marginal tax rates by assuming an increase in fixed percentage of income (we used a 3% increase for this purpose; sensitivity tests were performed by also computing marginal tax rates with a 0.1% shock instead, and the resulting figures were identical after winsorising the 1% lowest and largest values). We thus obtained from `EUROMOD`'s computations the marginal tax rates for capital income and labour income and the effective tax rate on total income, while the figures for labour, capital income and age (together with survey weights) are from the EU-SILC survey (although extracted from the `EUROMOD` database). An additional adjustment we make is to transform net labour income into gross income at production costs. The latter is needed because part of taxes and social insurance contributions are levied, in some countries, on employers and may or may not be deductible at the level of the employee's personal income tax. This grossing-up is obtained by simply adding up net income to all taxes and social insurance contributions paid. Therefore, tax rates are obtained in the end as shares of such gross wage (and, consequently, might look smaller compared to statutory rates).

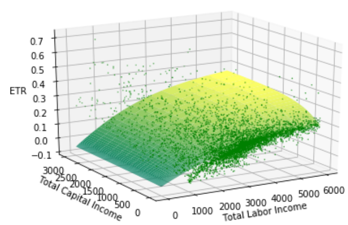
When we look at the raw output from the `EUROMOD` model, we find that there are several observations with extreme values for their effective tax rate. Since effective (marginal and average) rates are calculated as ratios, unrealistically large values might be obtained, for example when the denominator is a measure of income and this is very small. We omit such outliers by imposing the following restrictions upon the raw output of the microsimulation model. First, we exclude observations with an effective tax rate greater than 70% and observations with a marginal tax rate greater than 75% or less than 0%. Second, we drop observations from the microsimulation model where adjusted total income is less than 5 EUR. Because the tax rates are estimated as functions of income levels in the microdata, we have to adjust the model income units to match the units of the microdata. To do this, we find the factor such that factor times average steady-state model income equals the mean income in the final year of the microdata (for more details on the transforming factor see the last paragraph in section 2.3.1).

After having performed these adjustments, we fit tax functions by minimizing, alternatively, the sum of squared residuals, of absolute residuals, or by means of "robust" estimation methods. For the latter we employ the M-estimators Huber (Huber et al. (1973), Huber (1992)) and Bisquare (Beaton and Tukey (1974)), as illustrated in Maronna et al. (2018) and replicating the implementation of Jann (2010). We then select the method that offers the best compromise between Gaussian efficiency and representativeness of the data, particularly of salient features of the tax schedule like its progressivity. Because the data obtained from the `EUROMOD` microsimulation model are often heavily sparse in the space defined by labour income, capital income and tax rates, they are not always well approximated by the assumption of normally-distributed errors. Thus, M-estimators may occasionally offer higher efficiency, particularly in the form of lower sensitivity to very large values in the tails of the distribution.

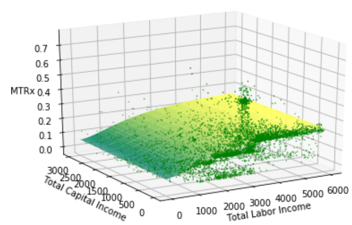
In the case of Italy we observe a structural break point around retirement age. This is because individuals who are yet non-retired mostly earn labour income (subject to the personal income tax) and capital income from rents and financial assets (which are largely subject to flat-rate taxation). On the other hand, retired individuals earn a large part of their earnings as capital income, which for them also includes pensions that are subject to the same progressivity as labour income. Thus, two separate sets of tax functions were estimated, respectively for a group of "young" individuals (up and including age 55) and for "old" individuals (from age 56 onward). We employed squared-error minimization to fit all these curves, except for the ETR of the "young" group, for which the robust M-estimator Bisquare was deemed the best choice as it significantly improves the representativeness of the estimated tax function with respect to labour income tax progressivity.

Figure 15 compares the estimated tax functions for ETR, $MTRx$ and $MTRy$ (this is represented by the surfaces in the Figures) against the corresponding data points, separately for the "young" and the "old" groups.

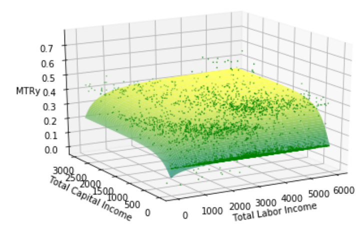
Figure 15: Estimated tax rate functions of ETR, $MTRx$, $MTRy$, by age group as functions of labour income and capital income from microsimulation model, year 2015



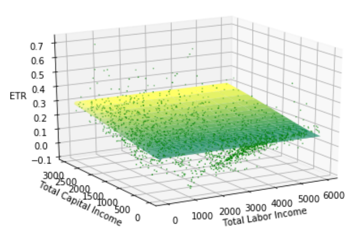
(a) Effective tax rates (ETR), "young" group



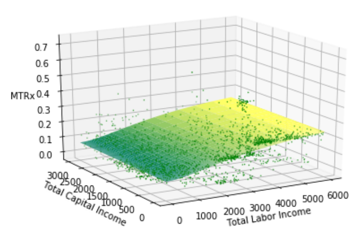
(b) Marginal tax rates on labour income ($MTRx$), "young" group



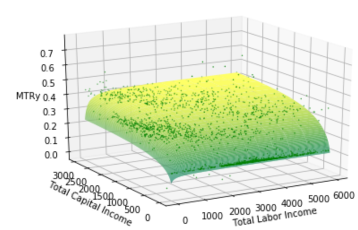
(c) Marginal tax rates on capital income ($MTRy$), "young" group



(d) Effective tax rates (ETR), "old" group



(e) Marginal tax rates on labour income ($MTRx$), "old" group



(f) Marginal tax rates on capital income ($MTRy$), "old" group

3.4.2 Social Insurance Contribution Functions

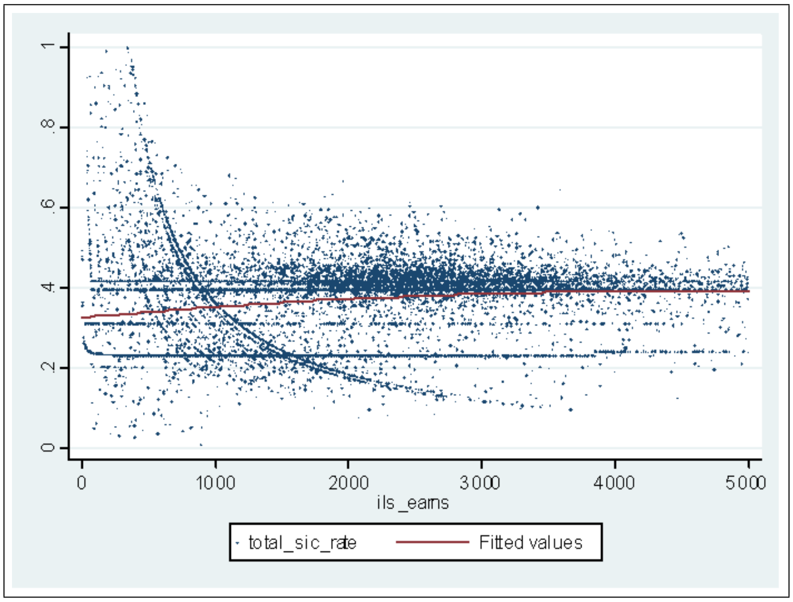
Here we detail the estimation strategy employed to represent social insurance contributions (SIC). The choice of considering mandatory contributions to pension schemes as something entirely distinct from income taxation is prompted by their nature of forced savings. Although we recognise that in most countries pension contributions include some degree of solidarity across income quantiles which makes part of the contribution akin to ordinary (redistributive) income taxation, we chose to keep the two separated for several reasons. First, this choice makes it easier to run policy simulations that only affect either taxation or SIC, as only one of the functions (for taxes or SIC) needs to be re-estimated. Second, running policy simulations is more transparent and free from confounding factors as any change in taxation (respectively, in SIC) will be simulated keeping SIC (respectively, tax rates) constant. Third, modelling-wise this choice better suits the need to compute pension benefits at the individual and aggregate levels.

In line with the estimation of tax functions (refer again to previous section 3.4.1) we define labour income as earned income, which is the sum of wages, salaries and self-employment income, expressed as gross wage at production costs (that is, before subtracting any tax or SIC). Because social insurance contributions are only levied on labour income and never on capital income, we do not need the more complex functional form used for tax functions and instead express SIC effective and marginal rates as function solely of labour income. Data on social contributions were obtained from the `EUROMOD` model which, as stated previously, employs EU-SILC survey data. From the `EUROMOD` we extracted the variables *sicer*, *sices* and *sicee* providing information on the amount of SIC paid respectively by employers, employees and self-employed workers. These three items are summed up to get the overall SIC, and dividing SIC by gross labour income we obtain a SIC rate. The SIC rate thus computed is regressed using OLS on gross labour income (GLI), its squared and cubic values:

$$SIC\,rate = \alpha_i + \beta_1 GLI_i + \beta_2 GLI_i^2 + \beta_3 GLI_i^3 + \varepsilon_i \tag{70}$$

By estimating a cubic polynomial we are able to capture common characteristics of the social security systems, namely the fact that SIC rates tend to be lower for low incomes (this is due to the existence of various allowances), then fairly constant for a large range of incomes, and then again lower for high incomes due to the existence of ceilings in contribution (see Figure 16). The marginal SIC rates are then derived employing the coefficients obtained from the OLS regression, computing SIC rates at the current level of income and at a 3% larger income, finally dividing the difference in SIC rate by the difference in incomes.

Figure 16: Scatter plot of social insurance contribution SIC rate data and the cubic polynomial regression line

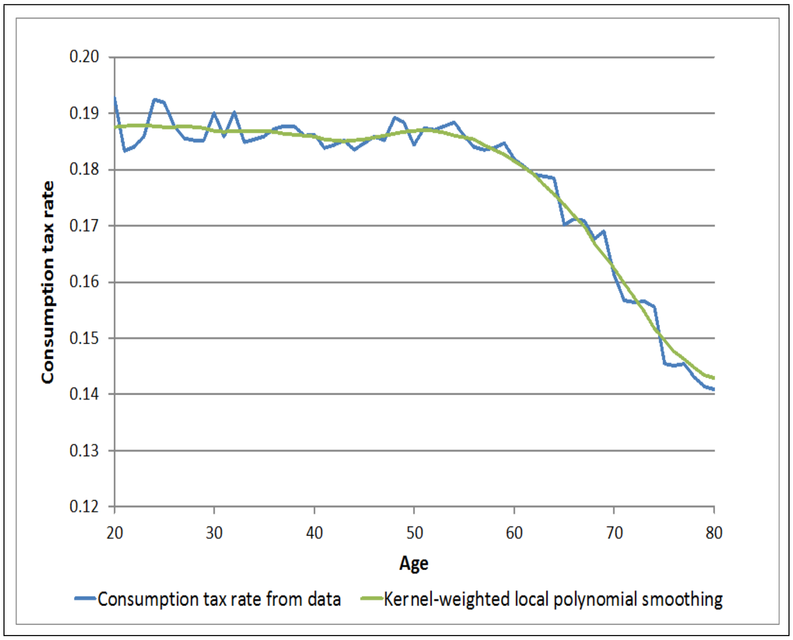


3.4.3 Consumption Tax Function

The consumption tax rate in `EDGE-M3` is differentiated by age of individuals. The consumption data are from the Italian Household Budget Survey (*Indagine sui Consumi delle Famiglie*), which is combined with the Italian Survey of Income and Living Conditions (IT-SILC) using statistical matching. The data and methodology are further described in Schez et al. (2016). The effective consumption tax is calculated as the total taxes paid divided by total expenditure. This is calculated by age of the household head, which gives the rates shown in Figure 17. The raw data has been smoothed using a Kernel-weighted local polynomial smoothing function, also shown. The smoothed values are entered into the model.

As shown, including an age-dimension to consumption tax allows us to capture the different rates faced by age-groups, in particular that older generations face progressively lower effective rates. This is due to the different average consumption baskets of different age groups. Sufficient data are only available up until age 80. For ages above 80, the rate is set at the 80-year old level. Lastly, note that the consumption variable in the model is exclusive of consumption tax, see for example equation (1).

Figure 17: Consumption tax data by age and smoothed function



4 Concluding Remarks

In this paper we presented EDGE-M3, the overlapping generations model for the EU countries. The model's richness makes it highly suitable for the analysis of the long-run impact of tax policy, pension policy and demographic changes on the economy as well as across and within generations. The model could potentially be extended to analyse social policies too. The model embeds 560 different types of individuals, i.e. 80 generations across seven income groups. Furthermore, EDGE-M3 includes detailed demographic trends provided by Eurostat and extensively uses microeconomic data for its parametrisation. A particularly interesting feature of the model are the non-linear income tax functions for labour and capital income as well as for social insurance contributions, estimated using the EUROMOD microsimulation model and the EU Statistics on Income and Living Conditions (EU-SILC). By means of these functions the richness of the underlying tax code is brought into the macro model. In particular, these functions allow to capture non-linearities such as local progressivity or regressivity of the schedule, and interactions between labour and capital income taxation. The close connection between EDGE-M3 and EUROMOD allows in particular the possibility to simulate policies with precision using EUROMOD and to analyse their macro-fiscal and redistributive impact using EDGE-M3 .

The EDGE-M3 model is a very suitable tool for running tax policy simulations or analysing the impact of the demographic change on the economy. For example, a change in the parameters of the income tax system could first be simulated with the microsimulation model EUROMOD . Then, the changed output from the microsimulation model can be readily employed in the EDGE-M3 macroeconomic model to study the dynamic and second-round effects of said reform. Consequently, `EDGE-M3` effectively combines both microeconomic and macroeconomic methods enabling a unified approach to policy analysis. An additional benefit for policy analysis is the model's ability to compute simulated transition paths from the current state to the steady-state equilibrium condition, which better informs about the short-run effects of a policy and its immediate impact on the public budget. The output of the model provides broken-down effects by household's age and income level, separately for each period of the transition path, thus enabling in-depth understanding of the evolution in time of the differential impact of a reform.

The `EDGE-M3` is currently being extended to analyse European pension systems, namely the defined benefit, the defined contribution and the point system. Thus, it will be possible to study the impact of switching from one pension system to another one on the whole economy as well as by different generations and income groups. This enables, among other things, studying pressing issues in the ageing debate, like the impact of a change in the payroll tax rate and/or an increase in the retirement age.

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Appendices

A Exogenous Inputs and Endogenous Output

A.1 Exogenous Parameters

All exogenous parameters that are inputs to the model are listed in Table 6. In addition, Table 7 lists the optional exogenous parameter inputs for the open economy version of the model.

A.2 Endogenous Variables

The endogenous variables of the `EDGE-M3` are listed in Table 8.

B Equilibrium Definitions and Solution Methods

In this section, we define the stationary steady-state and non-steady-state equilibrium of the `EDGE-M3` model. Sections 2.1 through 2.4 derive the equations that characterise the equilibrium of the model. However, we cannot solve for any equilibrium of the model in the presence of non-stationarity in the variables. Non-stationarity in `EDGE-M3` comes from productivity growth g_y in the production function (13) and population growth $\tilde{g}_{n,t}$ as described in section 3.1. We showed in section 2.5 how to stationarise all the characterising equations. To solve for the non-steady-state equilibrium transition path the steady-state solution is needed. As with the steady-state equilibrium, we must use the stationarised version of the characterising equations from Section 2.5.

B.1 Stationary Steady-state Equilibrium

With the stationarised model, we can now define the stationary steady-state equilibrium. This equilibrium will be long-run values of the endogenous variables that are constant over time. In a perfect foresight model, the steady-state equilibrium is the state of the economy at which the model settles after a finite amount of time, regardless of the initial condition of the model. Once the model arrives at the steady-state, it stays there indefinitely unless it receives some type of shock or stimulus.

These stationary values have all the growth components from productivity growth and population growth removed as defined in Table 2. Because the productivity growth rate g_y and population growth rate series $\tilde{g}_{n,t}$ are exogenous, we can transform the stationary equilibrium values of the variables back to their nonstationary values by reversing the identities in Table 2.

Definition 1 (Stationary steady-state equilibrium). A non-autarkic stationary steady-state equilibrium in the `EDGE-M3` model is defined as constant allocations of stationary household labour supply $\hat{n}_{j,s,t} = \bar{n}_{j,s}$ and savings $\hat{b}_{j,s+1,t+1} = \bar{b}_{j,s+1}$ for all j, t, and $E+1 \leq s \leq E+S$, and constant prices $\hat{w}_t = \bar{w}$ and $r_t = \bar{r}$ for all t such that the following conditions hold:

Table 6: List of exogenous parameters and baseline calibration values

Symbol	Description	Value
S	Maximum periods in economically active household life	80
E	Number of periods of youth economically outside the model	$\text{round}\left(\frac{S}{4}\right) = 20$
T_1	Number of periods to steady state for initial time path guesses	320
T_2	Maximum number of periods to steady state for nonsteady-state equilibrium	400
R	Eligible age for pension transfers	62
$\{\{\omega_{s,0}\}_{s=1}^{E+S}\}_{t=0}^{T_2+S-1}$	Initial population distribution by age	(see Sec. 3.1)
$\{f_s\}_{s=1}^{E+S}$	Fertility rates by age	(see Sec. 3.1.1)
$\{i_s\}_{s=1}^{E+S}$	Immigration rates by age	(see Sec. 3.1.2)
$\{\rho_s\}_{s=0}^{E+S}$	Mortality rates by age	(see Sec. 3.1.3)
$\{e_{j,s}\}_{j,s=1}^{J,S}$	Deterministic ability process	(see Sec. 3.2)
$\{\lambda_j\}_{j=1}^{J}$	Lifetime income group percentages	$[0.25, 0.25, 0.20, 0.10, 0.10, 0.09, 0.01]$
J	Number of lifetime income groups	7
$\tilde{l}$	Maximum labour supply	1
β	Discount factor	$(0.975)^{\frac{80}{S}}$
σ	Coefficient of constant relative risk aversion	2.2
b	Scale parameter in utility of leisure	0.527
υ	Shape parameter in utility of leisure	1.497
	Frisch elasticity	0.9
χ_s^n	Disutility of labour level parameters	(see Sec. 3.3.1)
χ_j^b	Utility of bequests level parameters	(see Sec. 3.3.2)
ζ	Share of bequests received by income-age households	(see Sec. 3.3.2)
μ	Share of government consumption in GDP	0.151
η	Share of transfers received by income-age households	(see Sec. 2.3.3)
τ_s^c	Marginal tax rate on consumption by age	(see Sec. 2.3.2)
Z	Level parameter in production function	1.0
γ	Capital share of income	0.4
δ	Capital depreciation rate	$1-(1-0.044)^{\frac{80}{S}} = 0.044$
g_y	Growth rate of labour augmenting technological progress	$(1+0.01)^{\frac{80}{S}} - 1 = 0.01$

Table 7: List of optional exogenous parameters and baseline calibration values

Symbol	Description	Value
r^*	World interest rate	0.0062
τ_d	Sovereign interest rate scale parameter	0.626
μ_d	Sovereign interest rate shift parameter	0.0036
ζ_K	Foreign share of capital	0.064
ζ_D	Foreign share of new debt issues	0.32

Table 8: List of endogenous variables

Symbol	Description
$n_{j,s}$	Labour supply by age and ability
$b_{j,s}$	Savings by age and ability
$c_{j,s}$	Consumption by age and ability
$T^I_{j,s}$	Total income tax by age and ability
$T^C_{j,s}$	Total consumption tax by age and ability
r	Real interest rate
w	Real wage rate
$\tau^{mtrx}_{j,s}$	Marginal tax rate on labour income by age and ability
$\tau^{mtry}_{j,s}$	Marginal tax rate on capital income by age and ability
$\tau^{etr}_{j,s}$	Average tax rate on income by age and ability
L	Aggregate labour supply
BQ	Aggregate bequests
K	Aggregate capital stock
K^d	Capital supplied domestically [Open economy option]
K^f	Capital supplied by foreign investors [Open economy option]
C	Aggregate consumption
I	Aggregate investment
Y	Aggregate output
TR	Aggregate transfers
D	Total public debt
D^d	Public debt held domestically [Open economy option]
D^f	Public debt held by foreign investors [Open economy option]
$factor_t$	Factor transforming model units into the data units

i. the population has reached its stationary steady-state distribution $\hat{\omega}_{s,t} = \bar{\omega}_s$ for all s and t as characterised in Section 3.1.4,

ii. households optimise according to (42), (43), and (43),

iii. firms optimise according to (47) and (16),

iv. markets clear according to (49), (50), and (52).

B.1.1 Stationary Steady-state Solution Method

This section describes the solution method for the stationary steady-state equilibrium described in Definition 1. The steady-state is characterised by $2JS$ equations and $2JS$ unknowns. However, because some of the other equations cannot be solved for analytically and substituted into the Euler equations, we must take a two-stage approach to the equilibrium solution. We first make a guess at steady-state interest rate $\bar{r}$, wage $\bar{w}$, and total bequests $\overline{BQ}$. We call these three steady-state guesses the "outer loop" of the steady-state solution method. They are the macroeconomic variables necessary to solve the household's problem.

The "inner loop" of the steady-state solution method is to solve for the steady-state household decisions $\bar{b}_{j,s}$ and labour supply $\bar{n}_{j,s}$ for all j and $E+1 \leq s \leq E+S$ given the values of the outer-loop variables. Because the lifetime optimisation problem of each household of type j is a highly non-linear system of $2S$ equations and $2S$ unknowns, we break the inner loop problem into two stages, the first of which is a univariate convex optimisation problem and the second of which is a serial series of univariate convex optimisation problems.

The first stage of the inner loop is to guess an initial steady-state consumption $\bar{c}_{j,E+1}$ for each household of type j. The second stage of the inner loop is to solve for each period household optimisation problem recursively given the initial consumption guess from the first stage. We update the first stage guess for $\bar{c}_{j,E+1}$ until the implied consumption in the last period $\bar{c}_{j,E+S}$ and savings in the last period $\bar{b}_{j,E+S+1}$ satisfy the last period savings Euler equation (43). We outline this algorithm in the following steps.

1. Use the techniques from Section 3.1.4 to solve for the steady-state population distribution vector $\bar{\boldsymbol{\omega}}$ and steady-state growth rate $\bar{g}_n$ of the exogenous population process.

2. Choose an initial guess for the values of the steady-state interest rate $\bar{r}^i$, wage $\bar{w}^i$, and total bequests $\overline{BQ}^i$, where superscript i is the index of the iteration number of the guess.

    (a) Note that if the production function is Cobb-Douglas ($\varepsilon = 1$), then you only have to guess the steady-state values of the steady-state interest rate $\bar{r}^i$, total bequests $\overline{BQ}^i$, total transfers $\overline{TR}^i$ and factor $\overline{factor}^i$. In this case, the steady-state wage $\bar{w}$ is determined by the interest rate using equation 48. In this Cobb-Douglas case ($\varepsilon = 1$), choosing both $\bar{r}$ and $\bar{w}$ in the outer loop can cause the solution method to not converge.

3. Given guesses for $\bar{r}^i$, $\overline{BQ}^i$, $\overline{TR}^i$ and $\overline{factor}^i$ and implied $\bar{w}^i$, solve for the steady-state household labour supply $\bar{n}_{j,s}$ and savings $\bar{b}_{j,s}$ decisions for all j and $E+1 \leq s \leq E+S$ using two-stage approach.

   (a) Given $\bar{r}^i$, $\bar{w}^i$, $\overline{BQ}^i$, $\overline{TR}^i$ and $\overline{factor}^i$, guess an initial steady-state consumption $\bar{c}^m_{j,E+1}$ for each type-j household, where m is the index of the inner-loop iteration.

      i. Given $\bar{r}^i$ and $\bar{w}^i$, $\overline{BQ}^i$, $\overline{TR}^i$, $\overline{factor}^i$, and $\bar{c}^m_{j,E+1}$, and the fact that $\bar{b}_{j,E+1} = 0$, we can use the household labour supply Euler equation (42) to solve for $\bar{n}_{j,E+1}$ for all j. This problem is a univariate root finder in $\bar{n}_{j,E+1}$.

$$\left(\bar{w}^i e_{j,s} - \frac{\partial \overline{T}^I_{j,s}}{\partial \bar{n}_{j,s}} - \frac{\partial \overline{T}^P_{j,s}}{\partial n_{j,s}}\right)(\bar{c}_{j,s})^{-\sigma}\left(\frac{1}{1+\bar{\tau}^c_s}\right) = e^{g_y(1-\sigma)}\chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_{j,s}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_{j,s}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}$$
$$\forall j, \quad \text{and} \quad E+1 \leq s \leq E+S$$

      ii. Given $\bar{c}^m_{j,E+1}$, $\bar{b}_{j,E+1} = 0$, and $\bar{n}_{j,E+1}$, we can use the household budget constraint (41) to solve analytically for $\bar{b}_{j,E+2}$ for all j.

$$\bar{b}_{j,s+1} = e^{-g_y}\left[(1+\bar{r}^i)\bar{b}_{j,s} + \bar{w}^i e_{j,s}\bar{n}_{j,s} + \zeta_{j,s}\frac{\overline{BQ}^i}{\lambda_j\bar{\omega}_s} + \eta_{j,s}\frac{\overline{TR}^i}{\lambda_j\bar{\omega}_s} - \overline{T}^I_{j,s} - \overline{T}^P_{j,s} - \bar{c}_{j,s}\left(1+\bar{\tau}^c_s\right)\right]$$
$$\forall j \quad \text{and} \quad E+1 \leq s \leq E+S$$

      iii. Given $\bar{c}^m_{j,E+1}$ and $\bar{b}_{j,E+2}$, use the household's $S-1$ dynamic Euler equations (43) to solve for $\bar{c}_{j,E+2}$ for all j. This problem is a univariate root finder in $\bar{c}_{j,E+2}$

$$(\bar{c}_{j,s})^{-\sigma} = e^{-\sigma g_y}\left[\chi^b_j\rho_s(\bar{b}_{j,s+1})^{-\sigma} + \beta\left(1-\rho_s\right)(\bar{c}_{j,s+1})^{-\sigma}\left(1+\bar{r}^i - \frac{\partial \overline{T}^I_{j,s+1}}{\partial \bar{b}_{j,s+1}}\right)\right]$$
$$\forall j, \quad \text{and} \quad E+1 \leq s \leq E+S-1$$

      iv. Repeat in serial steps (i) through (iii) until solved for the all households' steady-state lifetime decisions $\{\bar{c}_{j,s}, \bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ for all j.

   (b) Given household lifetime decisions $\{\bar{c}_{j,s}, \bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ for all j based on guesses for initial period consumption $\bar{c}^m_{j,E+1}$ for all j and outer loop guesses $\bar{r}^i$, $\overline{BQ}^i$, $\overline{TR}^i$, and $\overline{factor}^i$ check the error in the last period savings Euler equation (44) based on $\bar{c}_{j,E+S}$ and $\bar{b}_{j,E+S+1}$.

$$error_j \equiv e^{-\sigma g_y}\chi^b_j(\bar{b}_{j,E+S+1,t+1})^{-\sigma} - (\bar{c}_{j,E+S})^{-\sigma} \quad \forall j$$

   (c) If the error is greater than some small positive tolerance $error_j > toler_c$ for some j, then update the guesses for initial consumption $\bar{c}^{m+1}_{j,1}$ and repeat steps (a) and (b).

   (d) If the error is less than some small positive tolerance $error_j \leq toler_c$ for all j, then $\{\bar{c}_{j,s}, \bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ is the full set of partial equilibrium household steady-state solutions given guesses for $\bar{r}^i$, $\overline{BQ}^i$, $\overline{TR}^i$, and $\overline{factor}^i$.

4. Given partial equilibrium household steady-state solutions $\{\bar{c}_{j,s}, \bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ based on macroeconomic variable guesses $\bar{r}^i$, $\overline{BQ}^i$, $\overline{TR}^i$, and $\overline{factor}^i$ check the errors in the seven equations that characterise each of the macroeconomic variable guesses.

   (a) If we substitute the two market clearing conditions (49) and (50), and the firm's production function (45) into the firm's first order condition for capital demand (16), we get an expression in which household decisions $\{\bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ imply a value for the interest rate $\bar{r}^{i'}$.

$$\bar{r}^{i'} = \gamma\frac{\bar{Y}}{\bar{K}} - \delta$$
$$\text{where} \quad \bar{Y} = \bar{Z}(\bar{K})^{\gamma}(\bar{L})^{1-\gamma}$$
$$\text{and} \quad \bar{L} = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\bar{\omega}_s\lambda_j e_{j,s}\bar{n}_{j,s}$$
$$\text{and} \quad \bar{K} = \frac{1}{1+\bar{g}_n}\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\Big(\bar{\omega}_{s-1}\lambda_j\bar{b}_{j,s} + i_s\bar{\omega}_s\lambda_j\bar{b}_{j,s}\Big)$$

   The error for this variable is the percent difference between the initial guess for the interest rate $\bar{r}^i$ and the steady-state interest rate implied by household optimisation based on the initial guess $\bar{r}^{i'}$.

$$error_r = \frac{\bar{r}^{i'} - \bar{r}^i}{\bar{r}^i}$$

   (b) The stationarised law of motion for total bequests (52) provides the expression in which household decisions $\{\bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ imply a value for bequests $\overline{BQ}^{i'}$. Note that we need all the household decisions here because $\bar{r}^{i'}$ enters the equation on the right-hand-side.

$$\overline{BQ}^{i'} = \left(\frac{1+\bar{r}^{i'}}{1+\bar{g}_n}\right)\left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\rho_{s-1}\lambda_j\bar{\omega}_{s-1}\bar{b}_{j,s}\right)$$

   The error for this variable is the percent difference between the initial guess for total bequests $\overline{BQ}^i$ and the steady-state total bequests implied by household optimisation based on the initial guess $\overline{BQ}^{i'}$.

$$error_{bq} = \frac{\overline{BQ}^{i'} - \overline{BQ}^i}{\overline{BQ}^i}$$

   (c) The tax liability equation (25) as a function of stationarised labour income and stationarised capital income provides the expression in which household decisions $\{\bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ together with tax rates values imply a value for taxes paid $\overline{T}_{j,s,t}^{i'}$.

The stationarised law of motion for total transfers (53) implies a value of total transfers received $\overline{TR}^{i'}$.

$$\overline{TR}^{i'} = \left(\frac{1}{1+\bar{g}_n}\right)\left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\lambda_j\bar{\omega}_{s-1}\left[\tau^{etr}_{s,t}\left(\bar{w}_t e_{j,s}\bar{n}_{j,s,t} + \bar{r}_t\bar{b}_{j,s,t}\right)\right]\right)$$

The error for this variable is the percent difference between the initial guess for total transfers $\overline{TR}^{i}$ and the steady-state total transfers implied by household optimisation based on the initial guess $\overline{TR}^{i'}$.

$$error_{tr} = \frac{\overline{TR}^{i'} - \overline{TR}^{i}}{\overline{TR}^{i}}$$

(d) The factor equation (31) as a function of stationarised labour and capital income provides the expression in which household decisions $\{\bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ implies a value for average income in the model to be reconciled with its data counterpart by means of the $factor$.

$$factor\left[\sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\lambda_j\bar{\omega}_s\left(\bar{w}e_{j,s}\bar{n_{j,s}} + \bar{r}\bar{b_{j,s}}\right)\right] = \quad \text{average household income in data} \tag{71}$$

The error for this variable is the percent difference between the initial guess for the factor $\overline{factor}^{i}$ and the steady-state factor implied by household optimisation based on the initial guess $\overline{factor}^{i'}$.

$$error_{factor} = \frac{\overline{factor}^{i'} - \overline{factor}^{i}}{\overline{factor}^{i}}$$

5. If the maximum absolute error among the four outer loop error terms is greater than some small positive tolerance $toler_{out}$,

$$\max\left|\left(error_r, error_{bq}, error_{tr}, error_{factor}\right)\right| > toler_{out}$$

then update the guesses for the outer loop variables as a convex combination governed by $\xi_{ss} \in (0,1]$ of the respective initial guesses and the new implied values and repeat steps (3) through (5).

$$\left[\bar{r}^{i+1}, \overline{BQ}^{i+1}, \overline{TR}^{i+1}, \overline{factor}^{i+1}\right] = \xi_{ss}\left[\bar{r}^{i'}, \overline{BQ}^{i'}, \overline{TR}^{i'}, \overline{factor}^{i'}\right] + (1-\xi_{ss})\left[\bar{r}^{i}, \overline{BQ}^{i}, \overline{TR}^{i}, \overline{factor}^{i}\right]$$

6. If the maximum absolute error among the four outer loop error terms is less-than-or-equal-to some small positive tolerance $toler_{ss,out}$,

$$\max\left|\left(error_r, error_{bq}, error_{tr}, error_{factor}\right)\right| \leq toler_{ss,out}$$

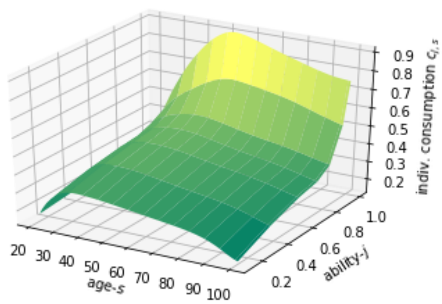
then the steady-state has been found.

(a) Make sure that the resource constraint (goods market clearing) (51) is satisfied. It is redundant, but this is a good check as to whether everything worked correctly.

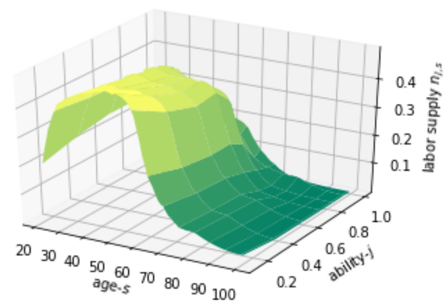
(b) Make sure that all the $2JS$ household Euler equations are solved to a satisfactory tolerance.

Figure 18 shows the household steady-state variables by age s and lifetime income group j using the baseline calibration described in Section 3.4.

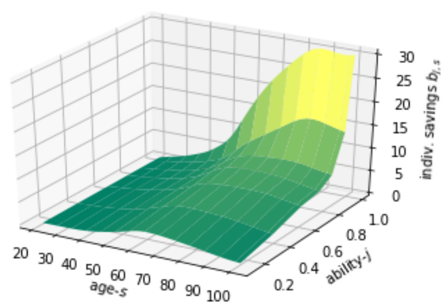
Figure 18: Steady-state distributions of household consumption $\bar{c}_{j,s}$, labour supply $\bar{n}_{j,s}$, and savings $\bar{b}_{j,s+1}$



(a) Consumption $\bar{c}_{j,s}$



(b) Labour supply $\bar{n}_{j,s}$



(c) Savings $\bar{b}_{j,s+1}$

Table 9 lists the steady-state prices and aggregate variable values along with some of the maximum error values from the characterising equations.

B.2 Stationary Non Steady-state Equilibrium

We define a stationary non-steady-state equilibrium as the following.

Table 9: Steady-state prices, aggregate variables, and maximum errors

Variable	Value	Variable	Value
$\bar{r}$	0.046	$\bar{w}$	1.624
$\bar{Y}$	0.603	$\bar{C}$	0.37
$\bar{I}$	0.145	$\bar{K}$	2.687
$\bar{L}$	0.223	$\bar{TR}$	0.109
$\overline{BQ}$	0.076		
Max. abs. labour supply Euler error	1.33e-13	Max. abs. savings Euler error	1.77e-13
Resource constraint error	-3.36e-11	Steady-state computation time	4 min. 4.6 sec.*

* The steady-state computation time does not include any of the exogenous parameter computation processes, e.g. the estimation of the baseline tax functions.

Definition 2 (**Stationary Non-steady-state functional equilibrium**)**.** A non autarkic nonsteady-state functional equilibrium in the `EDGE-M3` model is defined as stationary allocation functions of the state $\left\{n_{j,s,t} = \phi_s\big(\hat{\boldsymbol{\Gamma}}_t\big)\right\}_{s=E+1}^{E+S}$ and $\left\{\hat{b}_{j,s+1,t+1} = \psi_s\big(\hat{\boldsymbol{\Gamma}}_t\big)\right\}_{s=E+1}^{E+S}$ for all j and t and stationary price functions $\hat{w}(\hat{\boldsymbol{\Gamma}}_t)$ and $r(\hat{\boldsymbol{\Gamma}}_t)$ for all t such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterised in (11), and those beliefs about the future distribution of savings equal the realised outcome (rational expectations),

$$\hat{\boldsymbol{\Gamma}}_{t+u} = \hat{\boldsymbol{\Gamma}}^e_{t+u} = \Omega^u\left(\hat{\boldsymbol{\Gamma}}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimise according to (42), (43), and (43),

iii. firms optimise according to (47) and (16),

iv. markets clear according to (49), (50), and (52).

B.2.1 Stationary Non-steady-state Solution Method

This section describes the solution method for the stationary non-steady-state equilibrium described in Definition 2. We use the time path iteration (TPI) method. This method was originally outlined in a series of papers between 1981 and 1985[22] and in the seminal book Auerbach and Kotlikoff (1987, ch. 4) for the perfect foresight case and in Nishiyama and Smetters (2007, Appendix II) and Evans and Phillips (2014, Sec. 3.1) for the stochastic case. The intuition for the TPI solution method is that the economy is infinitely lived, even though the agents that make up the economy are not. Rather than recursively solving for

[22]See Auerbach et al. (1981, 1983), Auerbach and Kotlikoff (1983c,b,a), and Auerbach and Kotlikoff (1985).

equilibrium policy functions by iterating on individual value functions, one must recursively solve for the policy functions by iterating on the entire transition path of the endogenous objects in the economy (see Stokey et al. (1989, ch. 17)).

The key assumption is that the economy will reach the steady-state equilibrium $\bar{\boldsymbol{\Gamma}}$ described in Definition 1 in a finite number of periods $T < \infty$ regardless of the initial state $\hat{\boldsymbol{\Gamma}}_1$. The first step in solving for the non-steady-state equilibrium transition path is to solve for the steady-state using the method described in Section B.1.1. The next step is a transition path "outer loop" step, analogous to the outer loop described in the steady-state solution method. Guess transition paths for aggregate variables $\{\boldsymbol{r}^i, \boldsymbol{B\hat{Q}}^i, \boldsymbol{T\hat{R}}^i\}$, where $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$, $\boldsymbol{B\hat{Q}}^i = \left\{B\hat{Q}_1^i, B\hat{Q}_2^i, ...B\hat{Q}_T^i\right\}$, $\boldsymbol{T\hat{R}}^i = \left\{T\hat{R}_1^i, T\hat{R}_2^i, ...T\hat{R}_T^i\right\}$. The only requirement on these transition paths is that the initial total bequests $B\hat{Q}_1^i$ and initial total transfers $T\hat{R}_1^i$ conform to the initial state of the economy $\hat{\boldsymbol{\Gamma}}_1$, and that the economy has reached the steady-state by period $t = T$ $\{r_T^i, B\hat{Q}_T^i, T\hat{R}_T^i\} = \{\bar{r}, \overline{BQ}, \overline{TR}\}$.

The "inner loop" of the non-steady-state transition path solution method is to solve for the full set of lifetime savings decisions $\bar{b}_{j,s+1,t+1}$ and labour supply decisions $\bar{n}_{j,s,t}$ for every household that will be alive between periods $t = 1$ and $t = T$. Because we know the initial state of the economy $\hat{\boldsymbol{\Gamma}}_1$ in the transition path and we know the long-run steady-state $\bar{\boldsymbol{\Gamma}}$, we do not have to use the two-stage inner-loop method for solving the households' problems that we used in Section B.1.1. Because we know the neighbourhood where the solutions live, we can simply solve for the $2JS$ equations and unknowns for each household's lifetime decisions using a multivariate root finder. This is much faster than the two-stage method describe in Section B.1.1. We outline this algorithm in the following steps.

1. Compute the steady-state solution $\{\bar{n}_{j,s}, \bar{b}_{j,s}\}_{s=E+1}^{E+S}$ corresponding to Definition 1.

2. Given initial state of the economy $\hat{\boldsymbol{\Gamma}}_1$ and steady-state solutions $\{\bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$, guess transition paths of outer loop macroeconomic variables $\{\boldsymbol{r}^i, \boldsymbol{B\hat{Q}}^i, \boldsymbol{T\hat{R}}^i\}$ such that $B\hat{Q}_1^i$ and $T\hat{R}_1^i$ is consistent with $\hat{\boldsymbol{\Gamma}}_1$ and $\{r_t^i, B\hat{Q}_t^i, T\hat{R}_t^i\} = \{\bar{r}, \overline{BQ}, \overline{TR}\}$ for all $t \geq T_1$.

   (a) We choose two long-run time periods, T_1 and T_2. The first time period $t = T_1$ is the period in which the time paths of all the macroeconomic guesses hit their steady-state and stay at their steady-state thereafter. The second time period $t = T_2 > T_1$ is the period after which all the endogenous inner loop household variables hit their steady-state and stay at their steady-state thereafter. These two periods should be different because it requires time periods for the endogenous variables to hit the steady-state after the macroeconomic time path guesses have hit their steady-state.

3. Given initial condition $\hat{\boldsymbol{\Gamma}}_1$, outer-loop guesses for the aggregate time paths $\{\boldsymbol{r}^i, \boldsymbol{B\hat{Q}}^i, \boldsymbol{T\hat{R}}^i\}$, solve for the inner loop lifetime decisions of every household that will be alive across the time path $\{n_{j,s,t}, \hat{b}_{j,s+1,t+1}\}_{s=E+1}^{E+S}$ for all j and $1 \leq t \leq T_2$.

   (a) Given time path guesses $\{\boldsymbol{r}^i, \hat{\boldsymbol{w}}^i, \boldsymbol{B\hat{Q}}^i\}$, solve for each household's lifetime decisions $\{\hat{n}_{j,s,t}, \hat{b}_{j,s+1,t+1}\}_{s=E+1}^{E+S}$ for all j, $E+1 \leq s \leq E+S$, and $1 \leq t \leq T_2 + S - 1$.

i. In the transition path equilibrium solution method, the household problem can be solved with a multivariate root finder solving the $2S$ equations and unknowns at once for all j and $1 \leq t \leq T_2 + S - 1$, as opposed to the two-stage method for the steady-state solution described in Section B.1.1. Use $2S$ household Euler equations (42), (43), and (44) to solve for each household's $2S$ lifetime decisions.

ii. If one solves for each household's problem serially from the oldest households alive in period $t = 1$ to the youngest and then for every household born in period $t = 1, 2, ...T_2 - 1$, one can use the equilibrium guesses of the previous generation as initial guesses for the solver. This speeds up computation further and makes the initial guess for the highly nonlinear system of equations start closer to the solution value.

4. Given partial equilibrium household non-steady-state solutions $\{\hat{n}_{j,s,t}, \hat{b}_{j,s+1,t+1}\}_{s=E+1}^{E+S}$ for all j and $1 \leq t \leq T_2$ based on macroeconomic variable time path guesses $\{\hat{\boldsymbol{r}}^i, \boldsymbol{B\hat{Q}}^i, \boldsymbol{T\hat{R}}^i\}$, check the errors across the three time paths in the four equations that characterise each of the macroeconomic variable guesses.

(a) If we substitute the two market clearing conditions (49) and (50), and the firm's production function (45) into the firm's first order condition for capital demand (16), we get an expression in which household decisions $\{\hat{n}_{j,s,t}, \hat{b}_{j,s+1,t+1}\}_{s=E+1}^{E+S}$ imply the updated time path values for interest rate $r^{i'}$.

$$
\begin{aligned}
r_t^{i'} &= \gamma\frac{\hat{Y}_t}{\hat{K}_t} - \delta \\
&\text{where} \quad \hat{Y}_t = Z_t(\hat{K}_t)^\gamma(\hat{L}_t)^{1-\gamma} \\
&\text{and} \quad \hat{L}_t = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\hat{\omega}_{s,t}\lambda_j e_{j,s}\hat{n}_{j,s} \\
&\text{and} \quad \hat{K}_t = \frac{1}{1+\tilde{g}_{n,t}}\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\Big(\hat{\omega}_{s-1,t-1}\lambda_j\hat{b}_{j,s} + i_s\hat{\omega}_{s,t-1}\lambda_j\hat{b}_{j,s}\Big)
\end{aligned}
$$

The error for this variable is the percent difference between the initial guess for the interest rate path $\hat{r}^i$ and the updated path of interest rate implied by household optimisation based on the initial guess $\hat{r}^{i'}$.

$$
error_r = \frac{\hat{r}^{i'} - \hat{r}^i}{\hat{r}^i}
$$

(b) The stationarised law of motion for total bequests (52) provides the expression in which household decisions $\{\hat{n}_{j,s}, \hat{b}_{j,s+1}\}_{s=E+1}^{E+S}$ imply a value for bequests $B\hat{Q}^{i'}$. Note that we need all the household decisions here because $\bar{r}^{i'}$ enters the equation on the right-hand-side.

$$
B\hat{Q}^{i'} = \left(\frac{1+\hat{r}^{i'}}{1+\bar{g}_n}\right)\left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\rho_{s-1}\lambda_j\hat{\omega}_{s-1}\hat{b}_{j,s}\right)
$$

The error for this variable is the percent difference between the initial guess of the time path of total bequests $\hat{BQ}^{i}$ and the updated path of total bequests implied by household optimisation based on the initial guess $\hat{BQ}^{i'}$.

$$error_{bq} = \frac{\hat{BQ}^{i'} - \hat{BQ}^{i}}{\hat{BQ}^{i}}$$

(c) The tax liability equation (25) as a function of stationarised labour income and stationarised capital income provides the expression in which household decisions $\{\hat{n}_{j,s}, \hat{b}_{j,s+1}\}_{s=E+1}^{E+S}$ together with tax rates values imply the time path for taxes paid $\hat{T}^{i'}_{j,s,t}$. The stationarised law of motion for total transfers (53) implies the time path of total transfers received $\hat{TR}^{i'}$.

$$\hat{TR}^{i'} = \left(\frac{1}{1+\hat{g}_n}\right)\left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\lambda_j\hat{\omega}_{s-1}\left[\tau^{etr}_{s,t}\left(\hat{w}_t e_{j,s}\hat{n}_{j,s,t} + \hat{r}_t\hat{b}_{j,s,t}\right)\right]\right)$$

The error for this variable is the percent difference between the initial guess for total transfers $\hat{TR}^{i}$ and the steady-state total transfers implied by household optimisation based on the initial guess $\hat{TR}^{i'}$.

$$error_{tr} = \frac{\hat{TR}^{i'} - \hat{TR}^{i}}{\hat{TR}^{i}}$$

5. If the maximum absolute error among the four outer loop error terms is greater than some small positive tolerance $toler_{out}$,

$$\max\left|(error_r, error_{bq}, error_{tr}, error_{factor})\right| > toler_{out}$$

then update the guesses for the outer loop variables as a convex combination governed by $\xi_{tpi} \in (0,1]$ of the respective initial guesses and the new implied values and repeat steps (3) through (5).

$$\left[\hat{r}^{i+1}, \hat{BQ}^{i+1}, \hat{TR}^{i+1}\right] = \xi_{tpi}\left[\hat{r}^{i'}, hatBQ^{i'}, hatTR^{i'}\right] + (1-\xi_{tpi})\left[\hat{r}^{i}, hatBQ^{i}, hatTR^{i}\right]$$

6. If the maximum absolute error among the three outer loop error terms is less-than-or-equal-to some small positive tolerance $toler_{tpi,out}$,

$$\max\left|(error_r, error_{bq}, error_{tr})\right| \leq toler_{tpi,out}$$

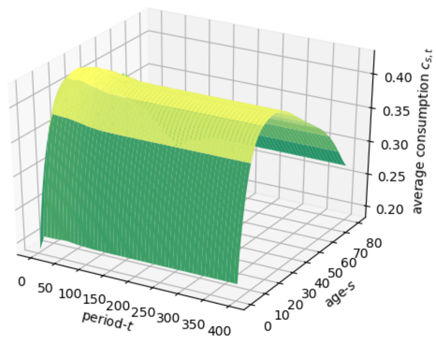
then a new steady-state has been found.

(a) Make sure that the resource constraint (goods market clearing) (51) is satisfied. It is redundant, but this is a good check as to whether everything worked correctly.

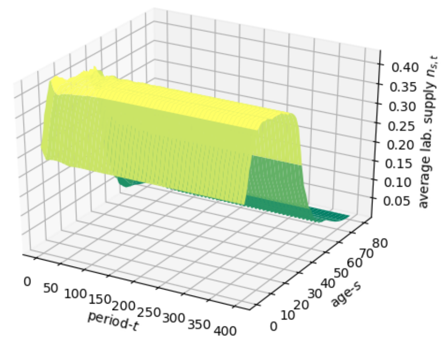
(b) Make sure that all the $2JS$ household Euler equations are solved to a satisfactory tolerance.

Figure 19 shows the household variables by age s oner time, averaged over lifetime income group j.

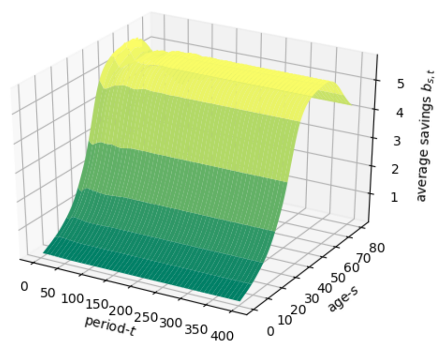
Figure 19: Non-steady-state distributions of household consumption $c_{s,t}$, labour supply $n_{s,t}$, and savings $b_{s+1,t+1}$



(a) Consumption $c_{s,t}$



(b) Labour supply $n_{s,t}$



(c) Savings $b_{s+1,t+1}$

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